

Finite temperature Thirring model: from linearization through canonical transformations to correct normal form of thermofield solution

V.V. Semenov and S.E. Korenblit (E-mail: korenb@ic.isu.ru)

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Abstract

It is shown that exact solvability of the finite temperature massless Thirring model, as well as of its zero temperature case, in canonical quantization scheme originates from the intrinsic hidden exact linearizability of Heisenberg equations by means of dynamical mapping onto the Schrödinger physical fields. The normal forms of different one- and two- parametric (thermo) field's solutions are obtained. They are connected with each other by making use of generalized conformal shift transformations. The sequential use of bosonic canonical transformations provides a correct renormalization, anticommutation and symmetry properties of these solutions. The dynamical role of inequivalent representations of 1+1-D free massless Dirac fields, that are induced by inequivalent representations of 1+1-D free massless (pseudo) scalar field, and the appearance of Schwinger terms are elucidated. The inequivalent vacuum is established as coherent state for $SU(1,1)$ group. A new alternative sources of superselection rules are shown. A generalization of Ojima tilde conjugation rules is suggested, which reveals the properties of coherent state for $SU(2)$ group for the fermionic thermal vacuum state and is useful for the thermofield bosonization. The notions of “hot” and “cold” thermofields are introduced to distinguish different thermofield representations giving the correct normal form of thermofield solution. The weak sense of definition of zero and finite temperature operator bosonization rules in the framework of thermofield dynamics is demonstrated.

1 Introduction

Despite a considerable age the two-dimensional Thirring model [1]–[3] is still remained as important touchstone for non-perturbative methods of quantum field theory [4]–[9] revealing new features both in the well-known [10]–[24] and in newly obtained solutions [25]–[29], [32]. At the same time the methods of integration of such two-dimensional models provide a clue for understanding some non-linear theories of higher dimensions [13]. In particular the Thirring model turns out to be a two-dimensional analog of the well-known Nambu-Jona-Lasinio model [13], [28, 29] and together with the Schwinger model provides an important example of using the well-known bosonization procedure (BP) [8]–[33], [34].

In the present work the BP for Thirring model is considered as a particular case of dynamical mapping (DM) [35], [36], what for Schwinger model was previously done in Greenberg's works [37]. In the framework of canonical quantization scheme [40] the DM method [35] consists of the construction of Heisenberg field (HF) $\Psi(x)$ as a solution of Heisenberg equations of motion (HEq) in the form of Haag expansion built as a sum of normal products [42] of free physical fields $\psi(x)$, whose representation space accords with unknown

a priori physical states of the given field theory [35]. The DM $\Psi(x) \stackrel{w}{=} \Upsilon[\psi(x)]$, being generally speaking a weak equality, implies the choice of appropriate initial conditions for the HEq. For example [33], [35], if both sets of fields are complete, irreducible and coincide asymptotically as $t \rightarrow -\infty$, the HF will tend in a weak sense to appropriate asymptotic physical field $\psi_{in}(x)$: $\lim_{t \rightarrow -\infty} \Psi(x^1, t) \stackrel{w}{=} \Upsilon[\psi_{in}(x^1, -\infty)]$. However the (asymptotic) completeness and irreducibility are absent in the presence of bound states [35], [36]. In particular for the exactly solvable two-dimensional Thirring and Schwinger models [13], [33] the physical asymptotic states of propagated physical particles have nothing to do with massless free Dirac asymptotic fields.

As was shown in the works [43]–[45], in general, it is more natural and convenient, in the spirit of Ref. [38, 39], to make DM onto the Schrödinger physical field $\psi_s(x)$, associated with the HF at $t \rightarrow 0$: $\lim_{t \rightarrow 0} \Psi(x^1, t) \stackrel{w}{=} \Upsilon[\psi_s(x^1, 0)]$, which is a generalization [44], [45] of the well-known interaction representation and is closely related to the procedure of canonical quantization [33], [40]. In this representation the corresponding time-dependent coefficient functions of DM [43], [44] contain all the information about bound states and scattering, and exactly solvable Federbush model [7] leads to the exactly linearizable HEq [45]. Such kind of exact linearization of HEq with so-generalized initial conditions in a weak sense allows to overcome the restrictions of Haag theorem [33, 40], removing them into the representation construction of the free physical field $\psi_s(x)$ [43, 44, 45, 54]. Arisen as Schrödinger physical field, the field $\psi_s(x)$ in fact plays the role of asymptotic one [41].

The present paper shows that HEq of the Thirring model admits a similar operator linearization and that the choice of free massless (pseudo) scalar fields as the physical ones is a consequence of reducibility of the free massless Dirac field [33] in the space of these fields. The problem of Schwinger terms in the currents commutator [4], being closely related to BP [10]–[33], also finds here a natural solution [45] in fact borrowed from QED [46], where it is also sufficient to define this commutator only for the free fields in the corresponding “interaction representation”.

The presence of finite temperature: $k_B T = 1/\varsigma > 0$, aggravates the above mentioned problems. The introduction of the temperature in quantum field theory may be achieved in the real-time formalism of Thermofield Dynamics [35], [48] by means of reordering with respect to a new thermal vacuum $|0(\varsigma)\rangle$ the above DM expression of the HF $\Psi(x) \stackrel{w}{=} \Upsilon[\psi(x)]$ given at first in terms of the free physical fields $\psi(x)$. The latter ones are defined now with respect to the “cold” vacuum $|0\tilde{0}\rangle$ at $T = 0$, which is connected with the new thermal vacuum $|0(\varsigma)\rangle$ by the thermal Bogoliubov transformation [35], [48]: $\mathcal{V}_\vartheta|0(\varsigma)\rangle = |0\tilde{0}\rangle$. Here for 1+1-D space-time: $\vartheta = \vartheta(k^1, \varsigma)$. Thus, for bosonic (B) or fermionic (F) fields with the energy $k^0 = \omega(k^1)$ one has, respectively:

$$|0(\varsigma)\rangle_{(B,F)} = \mathcal{V}_{\vartheta(B,F)}^{-1}|0\tilde{0}\rangle_{(B,F)}, \quad \text{where for: } g(k^1, \varsigma)\{\mp\}f(k^1, \varsigma) = 1, \quad (1)$$

$$\left. \begin{array}{l} (B)\{-\} : \quad \cosh^2 \vartheta(k^1, \varsigma) \\ (F)\{+\} : \quad \cos^2 \vartheta(k^1, \varsigma) \end{array} \right\} = g(k^1, \varsigma) = \exp\{\varsigma\omega(k^1)\}f(k^1, \varsigma). \quad (2)$$

If the corresponding physical thermofields $\psi(x, \varsigma)$ and interpolating them at $t \rightarrow \pm\infty$ thermofields $\Phi(x, \varsigma)$, as well as their tilde - conjugate ones $\tilde{\psi}(x, \varsigma)$ and $\tilde{\Phi}(x, \varsigma)$, are determined for $\sigma = \pm 1$ by the relations [35], (see however section 3):

$$\psi(x, \varsigma) = g^{1/2}(i\partial_1, \varsigma)\psi(x) - \sigma f^{1/2}(i\partial_1, \varsigma)\tilde{\psi}(x), \quad (3)$$

$$\Phi(x, \varsigma) = g^{1/2}(i\partial_1, \varsigma)\Psi(x) - \sigma f^{1/2}(i\partial_1, \varsigma)\tilde{\Psi}(x), \quad (4)$$

this means in general the absence of any equations of motion for these thermofields, since already the free fields $\psi(x)$ and $\tilde{\psi}(x)$ in Eq. (3) obey different free field equations [35]. The happy exceptions of this rule are at least the free Hermitian (pseudo) scalar field, $(\partial_\mu\partial^\mu + m^2)\varphi(x, \varsigma) = 0$, free massless Majorana field and the 1+1-D free massless Dirac field: $i(\gamma^\mu\partial_\mu)\chi(x, \varsigma) = 0$, with not yet important factor i .

However, the interaction contributions in the HEqs will inevitably destroy the superposition principle for the interpolating thermofield (4). Therefore, both for the self-interacting (pseudo) scalar field, with coinciding HEqs for the tilde HF $\tilde{\Psi}(x)$ and non-tilde HF $\Psi(x)$, and for a self-interacting Dirac field, where the above factor i now changes the relative sign of the interaction in the HEq for $\tilde{\Psi}(x)$ (see (196) below), the left hand side of the Eq. (4) is not a solution of the HEq for the fields in the right hand side.

However, assuming again both sets of fields are complete, irreducible [33] and asymptotically coincide at $t \rightarrow \pm\infty$, the Heisenberg fields $\Psi(x)$ and interpolating them thermofields (4) should tend in a weak sense [35] to the asymptotic physical fields $\psi(x)$ and physical thermofields (3) respectively, what is not the case for fermionic solutions of exactly solvable two-dimensional Thirring and Schwinger models [33].

Here, by using the methods of thermofield dynamics [35], [48], the answer to the question of [30] about the fate of operator bosonization relations for the Thirring model at finite temperature is given constructively. It is shown that HEq of Thirring model [1] admits exact solutions in terms of free (pseudo) scalar “cold” physical thermofields similar (but not the same!) to (3), as fermionic Heisenberg thermofields $\Psi(x, \varsigma)$, for which the relations like (4) are meaningless even asymptotically at $t \rightarrow \pm\infty$ in the weak sense [35], with respect to the both thermal (“hot”) $|0(\varsigma)\rangle_{(B,F)}$ or “cold” $|\tilde{0}\rangle_{(B,F)}$ vacua. To this end the DM onto the Schrödinger physical fields with above generalized weak initial condition at $t \rightarrow 0$ becomes especially relevant.

Unlike the recent works [51], [52], [53], by using direct step by step integration of exactly linearized HEq, we obtain the normal form of HF and Heisenberg thermofields, which includes all the necessary Klein factors providing a correct doubling of the number of degrees of freedom and consistent with all necessary anticommutation relations, renormalization, and symmetry conditions for the both free and Thirring thermofields.

To explain our method we start with zero temperature case in the next section, which extends and elaborates our previous results [54]. Definition of the model in canonical quantization scheme is given in the first subsection. Then the linearization procedure with corresponding definition of Heisenberg currents is advocated. The bosonization rules that we need for the free fields only are discussed in the subsection 3 with the appropriate choice of representation space for (pseudo) scalar fields. That all is used in subsection 4 for direct integration of HEq with chosen initial condition. The connection to different one- and two- parametric solutions, the values of Schwinger terms, and n - point fermionic Wightman functions are considered in subsection 5, where new sources of superselection rules are shown.

The third section is devoted to the finite temperature case and extends and elaborates our previous preliminary results [55], what was announced also in [56]. We start with qualitative thermodynamic manifestations of bosonization for the free massless 1D gases. Then we discuss an extension of Ojima rules. The notion of “hot” and “cold” thermofields is introduced in subsections 3.3, and in subsections 3.4 – 3.6 the thermal Thirring HEqs are obtained and solved by the same way as before, and the thermal n - point fermionic Wightman functions are obtained. The final remarks are made in Conclusion. Some details are explained, and useful formulas are collected in the Appendixes A-E.

2 Thirring model without temperature

2.1 Definition of model

Following to the canonical quantization procedure [40] we start with the formal Hamiltonian of the Thirring model [1], which in two-dimensional space-time¹ defines a Fermi self-interaction with fixed (and further unrenormalizable) dimensionless coupling constant g for spinor field with spin 1/2 and zero mass:

$$H[\Psi] = H_{0[\Psi]}(x^0) + H_{I[\Psi]}(x^0), \quad (5)$$

$$H_{I[\Psi]}(x^0) = \frac{g}{2} \int_{-\infty}^{\infty} dx^1 J_{(\Psi)\mu}(x) J_{(\Psi)}^{\mu}(x), \quad (6)$$

$$H_{0[\Psi]}(x^0) = \int_{-\infty}^{\infty} dx^1 \Psi^{\dagger}(x) E(P^1) \Psi(x), \quad E(P^1) = \gamma^5 P^1, \quad (7)$$

satisfying the equal-time canonical anticommutation relations (CAR) and locality condition:

$$\left\{ \Psi_{\xi}(x), \Psi_{\xi'}^{\dagger}(y) \right\} \Big|_{x^0=y^0} = \delta_{\xi,\xi'} \delta(x^1 - y^1), \quad (8)$$

$$\left\{ \Psi_{\xi}(x), \Psi_{\xi'}(y) \right\} \Big|_{x^0=y^0} = 0, \quad (9)$$

$$\left\{ \Psi_{\xi}(x), \Psi_{\xi'}^{\#}(y) \right\} \Big|_{(x-y)^2 < 0} = 0, \quad \text{with: } \Psi_{\xi}^{\#}(y) = \Psi_{\xi}(y), \quad \Psi_{\xi}^{\dagger}(y). \quad (10)$$

The indices $\xi, \xi' = \pm$, as well as for x^{ξ} , enumerate here the components of HF by the rule:

$$\Psi(x) = \begin{pmatrix} \Psi_1(x) \\ \Psi_2(x) \end{pmatrix} = \begin{pmatrix} \Psi_+(x) \\ \Psi_-(x) \end{pmatrix} \mapsto \Psi_{\xi}(x), \quad (\overline{\Psi}(x))_{\xi} = \Psi_{-\xi}^{\dagger}(x), \quad (11)$$

and the vector current $J_{(\Psi)}^{\mu}(x)$, together with the pseudovector current $J_{(\Psi)}^{5\mu}(x)$, for $\mu, \nu = 0, 1$, is their yet formal local bilinear functional of the form:

$$J_{(\Psi)}^{\mu}(x) \mapsto \overline{\Psi}(x) \gamma^{\mu} \Psi(x), \quad (12)$$

$$J_{(\Psi)}^{5\mu}(x) \mapsto \overline{\Psi}(x) \gamma^{\mu} \gamma^5 \Psi(x) = -\epsilon^{\mu\nu} J_{(\Psi)\nu}(x),$$

which due to (5)–(11) formally appears also in the canonical equations of motion² [3]–[7]:

$$i\partial_0 \Psi(x) = [\Psi(x), H[\Psi]] = \left[E(P^1) + g\gamma^0 \gamma_{\nu} J_{(\Psi)}^{\nu}(x) \right] \Psi(x), \quad (13)$$

$$\text{or: } 2\partial_{\xi} \Psi_{\xi}(x) = -ig J_{(\Psi)}^{-\xi}(x) \Psi_{\xi}(x), \quad \xi = \pm, \quad (14)$$

– for each ξ -component of the field (11), also are related formally to the corresponding current components:

$$J_{(\Psi)}^{\xi}(x) = J_{(\Psi)}^0(x) + \xi J_{(\Psi)}^1(x) \mapsto 2\Psi_{\xi}^{\dagger}(x) \Psi_{\xi}(x), \quad \xi = \pm. \quad (15)$$

The correct definitions of these formal operator products will be discussed hereinafter.

¹Here: $x^{\mu} = (x^0, x^1)$; $x^0 = t$; $\hbar = c = 1$; $\partial_{\mu} = (\partial_0, \partial_1)$; for $g^{\mu\nu}$: $g^{00} = -g^{11} = 1$; for $\epsilon^{\mu\nu}$: $\epsilon^{01} = -\epsilon^{10} = 1$; $\overline{\Psi}(x) = \Psi^{\dagger}(x) \gamma^0$; $\gamma^0 = \sigma_1$, $\gamma^1 = -i\sigma_2$, $\gamma^5 = \gamma^0 \gamma^1 = \sigma_3$, $\gamma^{\mu} \gamma^5 = -\epsilon^{\mu\nu} \gamma_{\nu}$, where σ_i – Pauli matrices, and I – unit matrix; $(px) = p^0 x^0 - p^1 x^1$, $x^{\xi} = x^0 + \xi x^1$, $P^1 = -i\partial_1 = -i\partial/\partial x^1$, $2\partial_{\xi} = 2\partial/\partial x^{\xi} = \partial_0 + \xi \partial_1$; summation over repeated $\xi = \pm$ is nowhere implied.

²Contribution to (13) due to non-commutativity of $J_{(\Psi)}^{\nu}(x)$ and $\Psi(x)$ is formally proportional to $\delta(0) \gamma^0 \gamma_{\nu} \gamma^0 \gamma^{\nu} = 0$.

2.2 Linearization of the Heisenberg equation

An immediate consequence of the field equations of motion (13), (14) are the local conservation laws [3]–[7] for the currents (12), (15):

$$\partial_\mu J_{(\Psi)}^\mu(x) = 0, \quad \partial_\mu J_{(\Psi)}^{5\mu}(x) = -\epsilon_{\mu\nu} \partial^\mu J_{(\Psi)}^\nu(x) = 0, \quad \text{or:} \quad \partial_\xi J_{(\Psi)}^\xi(x) = 0, \quad \xi = \pm, \quad (16)$$

fully determine their dynamics as a free one [4], [6]. Therefore it is not surprising, that by virtue of the same equations of motion (13), (14), as well as by means of the anti-commutation relations (8) for HF, it is a simple matter to show [45] that:

$$i\partial_0 \gamma^0 \gamma_\nu J_{(\Psi)}^\nu(x) - \left[\gamma^0 \gamma_\nu J_{(\Psi)}^\nu(x), H_{0[\Psi]}(x^0) \right] = iI \partial_\mu J_{(\Psi)}^\mu(x) + i\gamma^5 \epsilon_{\mu\nu} \partial^\mu J_{(\Psi)}^\nu(x) \equiv 0, \quad (17)$$

where the first term on the r.h.s. of equality (17) comes evidently from the left terms with $\nu = 0$, while the second term on the r.h.s. comes from the left terms with $\nu = 1$. The canonical equation of motion for this operator of “total current” in Eq. (13), containing of course its commutator with the total Hamiltonian $H[\Psi]$ (5), recasts then to the following equation:

$$i\partial_0 \gamma^0 \gamma_\nu J_{(\Psi)}^\nu(x) - \left[\gamma^0 \gamma_\nu J_{(\Psi)}^\nu(x), H_{0[\Psi]}(x^0) \right] = \left[\gamma^0 \gamma_\nu J_{(\Psi)}^\nu(x), H_{I[\Psi]}(x^0) \right] = 0, \quad (18)$$

which thus cannot contain a contribution from the commutator with the interaction Hamiltonian $H_{I[\Psi]}(x^0)$ given by Eq. (6). Hence, as well as for the Federbush model [45], a non-zero contribution of Schwinger terms in HEq (18) would be premature, because, due to Eq. (17), it leads to violation of the current conservation laws (16).

On the one hand, within the framework of canonical quantization procedure [40], the vanishing of expressions (17), (18) means, that temporal evolution of this “total current” is governed by a free Hamiltonian $H_{0[\chi]}(x^0)$ of the same form (7) quadratic on some kind of free massless trial physical Dirac fields $\chi(x)$, furnished by the same anti-commutation relations and by the same conservation laws for corresponding currents $J_{(\chi)}^\nu(x)$, $J_{(\chi)}^{5\nu}(x)$, defined formally by Eqs. (8)–(12), (15), (16) with $\Psi(x) \mapsto \chi(x)$:

$$i\partial_0 \gamma^0 \gamma_\nu J_{(\chi)}^\nu(x) - \left[\gamma^0 \gamma_\nu J_{(\chi)}^\nu(x), H_{0[\chi]}(x^0) \right] = iI \partial_\mu J_{(\chi)}^\mu(x) + i\gamma^5 \epsilon_{\mu\nu} \partial^\mu J_{(\chi)}^\nu(x) = 0. \quad (19)$$

On the other hand, the Heisenberg current operators appearing in (17), (18) acquire precise operator meaning – with non-vanishing Schwinger term – only after the choice of the representation space [4], [40], [49] for anticommutation relations (8)–(10) and subsequent reduction in this representation to the normal-ordered form by means of renormalization, for example, via point-splitting and subtraction of the vacuum expectation value [33]:

$$J_{(\Psi)}^0(x) \mapsto \lim_{\tilde{\varepsilon} \rightarrow 0} \hat{J}_{(\Psi)}^0(x; \tilde{\varepsilon}) = \hat{J}_{(\Psi)}^0(x), \quad J_{(\Psi)}^1(x) \mapsto \lim_{\tilde{\varepsilon} \rightarrow 0} \hat{J}_{(\Psi)}^1(x; \tilde{\varepsilon}) = \hat{J}_{(\Psi)}^1(x), \quad (20)$$

$$\text{where for: } \tilde{\varepsilon}^\mu = -\epsilon^{\mu\nu} \varepsilon_\nu, \quad \text{at first: } \tilde{\varepsilon}^0 = \varepsilon^1 \rightarrow 0, \quad \text{with fixed: } \tilde{\varepsilon}^1 = \varepsilon^0, \quad \varepsilon^2 = -\tilde{\varepsilon}^2 > 0, \quad (21)$$

$$\text{for: } \hat{J}_{(\Psi)}^\nu(x; a) = Z_{(\Psi)}^{-1}(a) \left[\bar{\Psi}(x+a) \gamma^\nu \Psi(x) - \langle 0 | \bar{\Psi}(x+a) \gamma^\nu \Psi(x) | 0 \rangle \right], \quad (22)$$

and accordingly for every ξ - component (15). The renormalization “constant” $Z_{(\Psi)}(a)$ is defined below in (91). The definition of renormalized current (20)–(22) used here corresponds to the well-known

Mandelstam-Schwinger prescription [9], [46] specified in the work [11] and, unlike Johnson definition [2], [7], directly depends on the choice of representation space via the vacuum expectation value [33] in Eq. (22) like the very meaning of Schwinger term [4, 11, 13]. One can show [11], these different current definitions for the massless case coincide only for the free Dirac fields (cf. Eqs. (25) and (83) below).

The comments given above jointly with the foregoing arguments deduced from Eq. (16)–(19) allow to identify in HEq (13), at least in a weak sense, the Heisenberg operator of “total current”, defined by Eqs. (12), (17), with that operator, defined by Eqs. (12), (19) for the free massless trial physical Dirac fields $\chi(x)$ and renormalized in the sense of normal form (20)–(22) up to an unknown yet constant β :

$$\gamma^0 \gamma_\nu J_\nu^\Psi(x) \stackrel{w}{\mapsto} \frac{\beta}{2\sqrt{\pi}} \gamma^0 \gamma_\nu \hat{J}_\nu^\chi(x), \quad \text{where:} \quad (23)$$

$$\hat{J}_\nu^\chi(x) = \lim_{\varepsilon, (\tilde{\varepsilon}) \rightarrow 0} \hat{J}_\nu^\chi(x; \varepsilon(\tilde{\varepsilon})) \equiv : J_\nu^\chi(x) :, \quad \text{for: } Z_\chi(a) = 1. \quad (24)$$

Here the symbol $: \dots :$ means for $Z_\chi(a) = 1$ the usual normal form [33, 42] with respect to free fermionic annihilation/creation operators of the field $\chi(x)$ [13], defined in Appendix E.

The identification (23), (24) leads to linearization of both Eqs. (13), (14) in the representation of these trial physical fields $\chi(x)$. Of course, the Eq. (13) is linearized with respect to x^0 , while the Eq. (14) – with respect to x^ξ . However, the latter equation is the preference of two-dimensional world with initial condition being far from evidence. Whereas the former equation admits the above-mentioned in the Introduction physically reasonable generalized initial condition at $t = x^0 = 0$: $\lim_{t \rightarrow 0} \Psi(x^1, t) \stackrel{w}{=} \Upsilon[\psi_s(x^1, 0)]$. The observed weak linearization (23) of HEq with so-generalized initial conditions in a weak sense allows below to overcome the restrictions of Haag theorem [33, 40], removing them into the representation construction of the massless free Dirac field $\chi(x)$. Since unlike [43, 44, 45] the massless free Dirac field does not defines here the true asymptotic states of the model, this initial condition does not fix here the constant β , which will be defined dynamically in subsequent subsections contrary to [4, 33].

2.3 Bosonization and scalar fields

As was shown in [45] such kind of linearization of HEq for the Federbush model directly leads to its solution in the form of DM $\Psi(x) = \Upsilon[\psi_1(x), \psi_2(x)]$ onto the free massive Dirac fields $\psi_{1,2}(x)$ with different non-zero masses $m_{1,2}$, or in the form of DM $\Psi(x) = \Upsilon[\Phi_1(x), \Phi_2(x)]$ onto the pseudoscalar sine-Gordon fields $\Phi_{1,2}(x)$ furnished by appropriately defined normal ordering prescription [7]. Unlike the massive case, the components $\chi_\xi(x)$ of two-dimensional free massless field become completely decoupled, $\partial_\xi \chi_\xi(x) = 0$. As a consequence, this field turns out to be essentially non-uniquely defined or reducible and equipped by many inequivalent representations both in the spaces of a free massless (pseudo) scalar field ($\phi(x)$), $\varphi(x)$ [33] and free massive pseudoscalar field $\phi_m(x)$ [25]. Of course, the DM is physically meaningful only onto the complete, irreducible sets of fields: $\Psi(x) = \Upsilon[\varphi(x), \phi(x)]$, or $\Psi(x) = \Upsilon[\phi_m(x)]$, or $\Psi(x) = \Upsilon[\psi_M(x)]$, – for the phase with spontaneously broken chiral symmetry [13]–[21], [29], so further we consider here only the first possibility. The corresponding DM is known also as BP, allows to operate with functionals of boson fields instead of fermionic operators and forms a powerful tool for obtaining non-perturbative solutions in various two-dimensional models [9], [10], [33], [13]. Its use also simplifies integration of the linearized HEq like (13) [45].

Being a formal consequence of the current conservation conditions (16) only, the bosonization rules have, generally speaking, the sense of weak equalities only for the current operator in the normal-ordered form (20)–(22), that already implies a choice of certain representations of (anti-) commutation relations (8) and (34) below. However, for the free massless fields $\chi(x)$, $\varphi(x)$, $\phi(x)$, this choice (364) and (46) below is carried out automatically. This, due to the linearization condition (23), (24), becomes enough for our purposes, since for the free fields these relationships appear as strong operator equalities [33, 58]:

$$\hat{J}_{(\chi)}^\mu(x) = \frac{1}{\sqrt{\pi}} \partial^\mu \varphi(x) = -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi(x), \quad \hat{J}_{(\chi)}^{-\xi}(x) = 2 : \chi_{-\xi}^\dagger(x) \chi_{-\xi}(x) := \frac{2}{\sqrt{\pi}} \partial_\xi \varphi^\xi(x^\xi), \quad (25)$$

$$\text{where: } a^\mu \partial_\mu \varphi^\xi(x^\xi) = a^\xi \partial_\xi \varphi^\xi(x^\xi) = a^\xi \partial_0 \varphi^\xi(x^\xi) = \xi a^\xi \partial_1 \varphi^\xi(x^\xi), \quad \text{for: } x^\xi = x^0 + \xi x^1, \quad (26)$$

$$\text{and for } \nu = 0, 1: \sum_{\xi=\pm} (\xi 1)^\nu \partial_\xi \varphi^\xi(x^\xi) = \partial_\nu \sum_{\xi=\pm} \varphi^\xi(x^\xi) = \epsilon_{\nu\mu} \partial^\mu \sum_{\xi=\pm} \xi \varphi^\xi(x^\xi). \quad (27)$$

Unlike [10] the free massless scalar and pseudoscalar fields $\varphi(x)$ and $\phi(x)$, $\partial_\mu \partial^\mu \varphi(x) = 0 = \partial_\mu \partial^\mu \phi(x)$, are taken mutually dual and coupled by symmetric integral relations with the step function $\varepsilon(x^1) = \text{sgn}(x^1)$:

$$\left. \begin{array}{l} \phi(x) \\ \varphi(x) \end{array} \right\} = -\frac{1}{2} \int_{-\infty}^{\infty} dy^1 \varepsilon(x^1 - y^1) \partial_0 \left\{ \begin{array}{l} \varphi(y^1, x^0) \\ \phi(y^1, x^0) \end{array} \right\}, \quad (28)$$

$$\text{where: } \varepsilon(x^1) = \frac{1}{i\pi} \int_{-\infty}^{\infty} dk^1 \text{P} \frac{1}{k^1} e^{ik^1 x^1} = \begin{cases} 1, & \text{for } x^1 > 0, \\ -1, & \text{for } x^1 < 0, \end{cases} \quad (29)$$

imposing the following asymptotical boundary conditions of solitonic type:

$$\varphi(-\infty, x^0) + \varphi(\infty, x^0) = 0, \quad \phi(-\infty, x^0) + \phi(\infty, x^0) = 0, \quad (30)$$

that admit also nonzero behavior on both space infinities $x^1 \rightarrow \pm\infty$, providing for the conserved charges of these fields, similar to [10, 13, 33, 34], usual and topological definitions (also with $\phi(x) \mapsto \hat{\phi}(x) + v_{cl}(x)$):

$$\left. \begin{array}{l} O \\ O_5 \end{array} \right\} = \lim_{L \rightarrow \infty} \int_{-\infty}^{\infty} dy^1 \Delta\left(\frac{y^1}{L}\right) \partial_0 \left\{ \begin{array}{l} \varphi(y^1, x^0) \\ \phi(y^1, x^0) \end{array} \right\} \xRightarrow{\Delta=1} \left\{ \begin{array}{l} \phi(-\infty, x^0) - \phi(\infty, x^0) \mapsto \hat{O} + o_{cl}, \\ \varphi(-\infty, x^0) - \varphi(\infty, x^0) \mapsto \hat{O}_5 + o_{5cl}. \end{array} \right. \quad (31)$$

The latter ones correspond to the possible c-number contribution of classical fields $v_{cl}(x)$, marking the various unitarily inequivalent vacua [33, 34] by various values of respective charges $o_{cl}, o_{5cl} \neq 0$ (31).

The above right ($\xi = -$) and left ($\xi = +$) moving fields $\varphi^\xi(x^\xi)$ and their charges Q^ξ for $\xi = \pm$ are defined by linear combinations [33]:

$$\varphi^\xi(x^\xi) = \frac{1}{2} [\varphi(x) - \xi \phi(x)], \quad Q^\xi = \frac{1}{2} [O - \xi O_5] \implies \xi \varphi^\xi(x^0 + \xi \infty) - \xi \varphi^\xi(x^0 - \xi \infty) = \pm 2 \varphi^\xi(x^0 \pm \infty), \quad (32)$$

$$\varphi(x) = \sum_{\xi=\pm} \varphi^\xi(x^\xi), \quad \phi(x) = -\sum_{\xi=\pm} \xi \varphi^\xi(x^\xi), \quad O = \sum_{\xi=\pm} Q^\xi, \quad O_5 = -\sum_{\xi=\pm} \xi Q^\xi. \quad (33)$$

The last expressions (32) for Q^ξ explicitly demonstrate the x^0 - independence of all charges. All commutation relations [10, 24, 33] for the quantum fields $\varphi(x)$, $\phi(x)$, $\varphi^\xi(x^\xi)$, and the quantum charges O, O_5, Q^ξ :

$$[\varphi(x), \partial_0 \varphi(y)]|_{x^0=y^0} = [\phi(x), \partial_0 \phi(y)]|_{x^0=y^0} = i\delta(x^1 - y^1), \quad (34)$$

$$[\varphi(x), \varphi(y)] = [\phi(x), \phi(y)] = -\frac{i}{2}\varepsilon(x^0 - y^0)\theta\left((x - y)^2\right) = \frac{1}{i}D_0(x - y), \quad (35)$$

$$[\varphi(x), \phi(y)] = \frac{i}{2}\varepsilon(x^1 - y^1)\theta\left(-(x - y)^2\right), \quad (36)$$

$$[\varphi(x), O] = [\phi(x), O_5] = i, \quad [\varphi(x), O_5] = [\phi(x), O] = 0, \quad (37)$$

$$[\varphi^\xi(s), \varphi^{\xi'}(\tau)] = -\frac{i}{4}\varepsilon(s - \tau)\delta_{\xi, \xi'}, \quad [\varphi^\xi(s), Q^{\xi'}] = \frac{i}{2}\delta_{\xi, \xi'}, \quad (38)$$

with the step function $\theta(s) = (1 + \varepsilon(s))/2$, are reproduced by commutators of their annihilation/creation (frequency) parts (see (344)–(347) of Appendix B) [6, 10, 13]:

$$\langle 0|\phi(x)\phi(y)|0\rangle = \langle 0|\varphi(x)\varphi(y)|0\rangle = [\phi^{(+)}(x), \phi^{(-)}(y)] = [\varphi^{(+)}(x), \varphi^{(-)}(y)] = \frac{1}{i}D^{(-)}(x - y) \equiv \quad (39)$$

$$\equiv \frac{1}{i}D^{(-)}(x^\xi - y^\xi) + \frac{1}{i}D^{(-)}(x^{-\xi} - y^{-\xi}) = -\frac{1}{4\pi}\ln\left(-\bar{\mu}^2(x - y)^2 + i0\varepsilon(x^0 - y^0)\right), \quad (40)$$

$$[\varphi^{\xi(\pm)}(s), \varphi^{\xi'(\mp)}(\tau)] = \pm\frac{\delta_{\xi, \xi'}}{i}D^{(-)}(\pm(s - \tau)) \equiv \mp\frac{\delta_{\xi, \xi'}}{4\pi}\ln\left(i\bar{\mu}\{\pm(s - \tau) - i0\}\right), \quad (41)$$

$$[\varphi^{\xi(\pm)}(s), Q^{\xi'(\mp)}] = \frac{i}{4}\delta_{\xi, \xi'}, \quad [Q^{\xi(\pm)}, Q^{\xi'(\mp)}] = \pm a_0\delta_{\xi, \xi'}, \quad [O^{(\pm)}, O^{(\mp)}] = [O_5^{(\pm)}, O_5^{(\mp)}] = \pm 2a_0, \quad (42)$$

$$a_0 = a_0(L) = \pi \int_0^\infty dk^1 k^0 \left(\delta_L(k^1)\right)^2, \quad L \rightarrow \infty, \quad \lim_{L \rightarrow \infty} \delta_L(k^1) = \delta(k^1), \quad (43)$$

defined here through the annihilation/creation operators $c(k^1)$, $c^\dagger(k^1)$ of the **pseudoscalar field** $\phi(x)$, integrated with some distributions [57]:

$$\mathcal{P}c(k^1)\mathcal{P}^{-1} = -c(-k^1), \quad [c(k^1), c^\dagger(q^1)] = 2\pi 2k^0\delta(k^1 - q^1), \quad c(k^1)|0\rangle = 0, \quad (44)$$

$$\text{for: } k^0 = |k^1|, \quad \mathcal{P}\frac{1}{k^1} = \frac{\varepsilon(k^1)}{|k^1|}, \quad \frac{1}{4}\left(\mathcal{P}\frac{1}{k^1} - \frac{\xi}{k^0}\right) = \frac{-\xi\theta(-\xi k^1)}{2k^0}, \quad \text{as:} \quad (45)$$

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{dk^1}{2k^0} \left[c(k^1) e^{-i(kx)} + c^\dagger(k^1) e^{i(kx)} \right] \equiv \phi^{(+)}(x) + \phi^{(-)}(x), \quad (46)$$

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^\infty dk^1 \mathcal{P}\frac{1}{k^1} \left[c(k^1) e^{-i(kx)} + c^\dagger(k^1) e^{i(kx)} \right] \equiv \varphi^{(+)}(x) + \varphi^{(-)}(x), \quad (47)$$

$$\varphi^{\xi(+)}(s) = -\frac{\xi}{2\pi} \int_{-\infty}^\infty \frac{dk^1}{2k^0} \theta(-\xi k^1) c(k^1) e^{-ik^0 s}, \quad (48)$$

$$\varphi^{\xi(-)}(s) = \left[\varphi^{\xi(+)}(s) \right]^\dagger, \quad \varphi^\xi(s) = \varphi^{\xi(+)}(s) + \varphi^{\xi(-)}(s), \quad (49)$$

$$Q^{\xi(+)}(\hat{x}^0) = \lim_{L \rightarrow \infty} \frac{i\xi}{2} \int_{-\infty}^{\infty} dk^1 \theta(-\xi k^1) c(k^1) e^{-ik^0 \hat{x}^0} \delta_L(k^1), \quad (50)$$

$$Q^{\xi(-)} = [Q^{\xi(+)}]^\dagger, \quad Q^\xi = Q^{\xi(+)} + Q^{\xi(-)}, \quad O_5 = \frac{i}{2} [c^\dagger(0) - c(0)]. \quad (51)$$

Here the expression (47) for the scalar field follows from substitution of Eqs. (46), (29) to the second line of Eq. (28), and, without loss of generality, here and below we omit the classical parts of fields and charges $v_{cl}(x), o_{cl}, o_{5cl}$ in the Eq. (31). The labeled by them inequivalent representations of (pseudo) scalar fields have a purely topological nature [33, 34], which is not relevant for our further dynamical purposes. According to Eqs. (50), (43), the quantum charges O, O_5, Q^ξ are defined by zero mode's contribution only. Nevertheless, the both definitions of Eq. (31) recast $\hat{O}_5 \Rightarrow O_5$ to the last formal expression of Eq. (51). Whereas for O, Q^ξ the distributions $\varepsilon(k^1), \theta(k^1)$ appear, inventing the infrared regularization (50), (43), which is necessary also to obtain all the relations (42).

According to [13], [24], the invariance under the parity transformation $\mathcal{P}\{\dots\}\mathcal{P}^{-1}$ of generating functional $\mathcal{Z}[\phi(x); \mathcal{J}]$ for the free massless pseudoscalar field $\phi(x)$ (46), (44), unlike that for the theory of scalar field $\varphi(x)$ (47), implies the following feature of the classical source $\mathcal{J}(x)$: $\mathcal{J}(-x^1, x^0) = -\mathcal{J}(x^1, x^0)$, $\int d^2x \mathcal{J}(x) = 0$, which automatically removes the zero mode contribution of this field into the generating functional, leading to its well definiteness and gauge invariance under field-translation by arbitrary constant. The appearance of Cauchy main value (45) in the expression (47) for the dual scalar field also automatically excludes its zero mode contribution [4]. This relaxes for the chosen representation space of pseudoscalar field (44) the problem of non-positivity of inner product [7, 33] induced by Wightman functions $D^{(-)}$ (39) or $D^{(-)}$ (41).

A number of alternative methods are used to solve this problem³. Some of them use instead of the state $|0\rangle$ of Eq. (44) the vacuum state $|\hat{v}\rangle$ averaged with respect to above gauge group (cf. (52) below), that provides by GNS construction for non-normalizable vacuum functional [33, 34] or by quantization in the indefinite metric space [26, 27, 33]. Another methods use the non degenerate vacuum state $|0\rangle$ of Eq. (44), taken or as the unique [8]–[10] or as the only one of degenerate family states [13]–[24, 29], or with or without spontaneous breaking of the above field-translation symmetry, or with [13]–[24, 29] or without [10] replacing of the infrared regularization parameter $\bar{\mu}$ (40), (41), (329) to arbitrary fixed finite scale M .

These various representation spaces of massless (pseudo) scalar field result in particular in various meanings of non-negative value a_0 (43) in Eq. (42) varying from 0 for some ones to ∞ for another ones. It depends in turn on the choice of the volume cut-off regularization function $\Delta(y^1/L)$, with the Fourier image $\delta_L(k^1)$ (243) (analysed in Appendix C), which should be inserted into the integral (31) for correct charge definition. Unfortunately any of such (continuous or not) cut-off regularization induces the fictitious and non-physical \hat{x}^0 - dependence of formally conserved charges O, O_5 (31) or Q^ξ (32) and their frequency parts (50), and, in general, destroys the last equalities of Eqs. (31), (32), important in topological sense [24, 25, 33, 58]. Such kind of \hat{x}^0 - dependence, being an artifact of the charge's regularization (31), should be eliminated at the end of calculation. Due to (43) this is achieved by virtue of thermodynamic limit $L \rightarrow \infty$. Indeed, because for zero temperature \hat{x}^0 does not appear explicitly in commutation relations (42), further without loss of generality it may be fixed as $\hat{x}^0 = 0$ in accordance with (31), (32). However,

³May be the most global of them is bosonization (DM) onto the free massive pseudoscalar field $\phi_m(x)$ [25] for $\beta^2 < 4\pi$, what eliminates this problem together with gauge invariance under field-translation.

for the nonzero temperature (238) this dependence will require additional care.

The parameter a_0 itself makes sense of regularization parameter, which in the end of calculation should disappear in physical quantities. This will be especially important in the next section, where it acquires a temperature dependence (239), (245). Besides, this parameter, for any real v , defines the vacuum expectation value (VEV) of the operator of the above field-translation gauge transformation:

$$\exp \{ivO_5\} \phi(x) \exp \{-ivO_5\} = \phi(x) + v, \quad (52)$$

$$\langle 0 | \exp \{ivO_5\} | 0 \rangle \equiv \langle 0 | v \rangle = \exp \left\{ -a_0 v^2 \right\}, \quad (53)$$

which is well known in quantum theory of free massless (pseudo) scalar field [47] §11.2.2. The state $|v\rangle$ is a well known coherent state of harmonic oscillator, which corresponds to zero mode like (51), $k^1 = 0$, for $L \rightarrow \infty$, and if simultaneously $a_0 \rightarrow \infty$, then $\langle 0 | v \rangle \rightarrow 0$, and the state $|v\rangle$ defines another orthogonal vacuum state of the above degenerate family [24, 35, 47], for example, for the case of usual box of the length $2L$ from the last but one column of the Table given in Appendix C. While for any continuous regularization function from another columns of this Table the finite a_0 means, that the charge definition has nothing to do with previous standard thermodynamic limit, and corresponds to another vacuum structure of the representation space of (pseudo) scalar field [13]–[24]. The randomization [24] of the function $\Delta(y^1/L)$ (or $k^0 \delta_L(k^1)$) and the corresponding value of a_0 may be used for infrared “stabilization” of Wightman functions (39), (41), which replaces [13] the infrared regularization parameter $\bar{\mu}$ (40), (41), (329) to an arbitrary finite scale M .

According to [6, 33, 34], in such a (more or less) well-defined space of bosonic fields (28)–(50) one can construct the variety of different inequivalent representations of the solutions of Dirac equation $\partial_\xi \chi_\xi(x) = 0$ for the free massless trial physical field $\chi(x)$ in the form of local normal-ordered exponentials of right and left moving boson fields $\varphi^{-\xi}(x^{-\xi})$ and their charges Q^ξ (32), (38). The most simple field [33], which satisfies to extra free equations of [6]: $\partial_s \chi_\xi(s) = -i\pi \mathcal{N}_\varphi \left\{ \hat{J}_{(\chi)}^\xi(s) \chi_\xi(s) \right\}$ with $s = x^{-\xi}$, and leads for $Z_{(\chi)}(a) = 1$ exactly to the bosonization relations (25) for the corresponding currents (20)–(22), reads⁴:

$$\chi_\xi(x) = \chi_\xi(x^{-\xi}) = \mathcal{N}_\varphi \left\{ \exp \left(-i2\sqrt{\pi} \varphi^{-\xi}(x^{-\xi}) \right) \right\} \exp \left(-i\sqrt{\pi} \frac{\xi}{2} Q^\xi \right) \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} e^{i\varpi - i\xi\Theta/4}, \quad (54)$$

$$\text{or: } \chi_\xi(x^{-\xi}) = \mathcal{N}_\varphi \left\{ \exp \left(-i2\sqrt{\pi} \left[\varphi^{-\xi}(x^{-\xi}) + \frac{\xi}{4} Q^\xi \right] \right) \right\} u_\xi, \quad (55)$$

$$\text{where: } u_\xi = \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -a_0 \frac{\pi}{8} \right\} \equiv u_\xi^{Ok} \exp \left\{ -a_0 \frac{\pi}{8} \right\}, \quad (56)$$

– is a two-component c-number spinor containing arbitrary, random, non-observable initial overall and relative phases ϖ and Θ and above mentioned regularization parameter a_0 from (42), (43), and the VEV (53). The infrared regularization parameter $\bar{\mu}$ from (41) can subsequently tend to zero [33] or remain to be fixed, $\bar{\mu} \mapsto M$, [13]–[24], depending on the phase of the model under consideration. The total field $\chi(x)$ is the sum of Right and Left fields defined according to their chirality, as $\gamma^5 \psi_R = +\psi_R$, $\gamma^5 \psi_L = -\psi_L$:

$$\chi(x) = \chi_R(x^-) + \chi_L(x^+) = \begin{pmatrix} \chi_+(x^-) \\ \chi_-(x^+) \end{pmatrix}, \quad \chi_R(x^-) = \begin{pmatrix} \chi_+(x^-) \\ 0 \end{pmatrix}, \quad \chi_L(x^+) = \begin{pmatrix} 0 \\ \chi_-(x^+) \end{pmatrix}. \quad (57)$$

⁴With taking into account the difference in $-\sqrt{2}$ of the field’s and charge’s normalization, and the losted coefficient 1/4 instead 1/2 before the charge in (11.94) of [33].

In accordance with (364) of Appendix E, the field $\chi_R(x^-)$ describes the right moving particles, while the $\chi_L(x^+)$ describes the left moving ones [33, 60, 61].

2.4 Integration of the Heisenberg equation

For the chosen representation (25)–(41) the operator product in the linearized by means of (23), (24) HEq (13) or (14) is naturally redefined into the normal-ordered form [33] with respect to the fields $\varphi^\xi(x^\xi)$:

$$\partial_0 \Psi_\xi(x) = \left(-\xi \partial_1 - i \frac{\beta g}{2\sqrt{\pi}} \hat{J}_{(\chi)}^{-\xi(-)}(x) \right) \Psi_\xi(x) - \Psi_\xi(x) \left(i \frac{\beta g}{2\sqrt{\pi}} \hat{J}_{(\chi)}^{-\xi(+)}(x) \right). \quad (58)$$

The famous expression for the derivative of function $F(x^1)$ in terms of the operator P^1 : $-i\partial_1 F(x^1) = [P^1, F(x^1)]$, and its finite-shift equivalent: $e^{iaP^1} F(x^1) e^{-iaP^1} = F(x^1 + a)$, allows to transcribe the equation (58) for $x^0 = t$, $\Psi_\xi(x) \longleftrightarrow Y(t)$, as follows:

$$\frac{d}{dt} Y(t) = A(t) Y(t) - Y(t) B(t), \quad (59)$$

and to obtain then its formal solution in the form of time-ordered exponentials:

$$Y(t) = T_A \left\{ \exp \left(\int_0^t d\tau A(\tau) \right) \right\} Y(0) \left[T_B \left\{ \exp \left(\int_0^t d\tau B(\tau) \right) \right\} \right]^{-1}, \quad (60)$$

that are immediately replaced here by the usual ones, recasting the solution already into the normal form:

$$\Psi_\xi(x) = e^{C^{\xi(-)}(x)} \Psi_\xi(x^1 - \xi x^0, 0) e^{C^{\xi(+)}(x)}, \quad (61)$$

where operator bosonization (25) of the vector current of trial field $\chi(x)$ (55) gives:

$$\begin{aligned} C^{\xi(\pm)}(x) &= -i \frac{\beta g}{2\sqrt{\pi}} \int_0^{x^0} dy^0 \hat{J}_{(\chi)}^{-\xi(\pm)}(x^1 + \xi y^0 - \xi x^0, y^0) = \\ &= -i \frac{\beta g}{2\pi} \left[\varphi^{(\pm)}(x^1, x^0) - \varphi^{(\pm)}(x^1 - \xi x^0, 0) \right] = -i \frac{\beta g}{2\pi} \left[\varphi^{\xi(\pm)}(x^\xi) - \varphi^{\xi(\pm)}(-x^{-\xi}) \right]. \end{aligned} \quad (62)$$

Remarkably, that the completely unknown “initial” HF $\Psi_\xi(x^1 - \xi x^0, 0) = \lambda_\xi(x^{-\xi})$ appears here also as a solution of free massless Dirac equation, $\partial_\xi \lambda_\xi(x^{-\xi}) = 0$, but certainly unitarily inequivalent to the free field $\chi(x)$ (55). The expressions (61), (62) suggest to choose it also in the normal-ordered form with respect to the field $\varphi^\xi(s)$ using appropriate bosonic canonical transformation of this field with constant parameters $\bar{\alpha} = 2\sqrt{\pi} \cosh \eta$ and $\bar{\beta} = 2\sqrt{\pi} \sinh \eta$, obeying $\bar{\alpha}^2 - \bar{\beta}^2 = 4\pi$, which is generated by the operator F_η (for $y^0 = x^0$) in the form $U_\eta = \exp F_\eta$:

$$U_\eta^{-1} \varphi^\xi(x^\xi) U_\eta = \omega^\xi(x^\xi) = \frac{1}{2\sqrt{\pi}} \left[\bar{\alpha} \varphi^\xi(x^\xi) + \bar{\beta} \varphi^{-\xi}(-x^\xi) \right], \quad (63)$$

$$U_\eta^{-1} \varphi(x) U_\eta = \omega(x) \equiv \omega^\xi(x^\xi) + \omega^{-\xi}(x^{-\xi}) = \frac{1}{2\sqrt{\pi}} \left[\bar{\alpha} \varphi(x^1, x^0) + \bar{\beta} \varphi(x^1, -x^0) \right], \quad (64)$$

$$U_\eta^{-1} \phi(x) U_\eta = \Omega(x) \equiv \xi \left(\omega^{-\xi}(x^{-\xi}) - \omega^\xi(x^\xi) \right) = \frac{1}{2\sqrt{\pi}} [\bar{\alpha}\phi(x^1, x^0) - \bar{\beta}\phi(x^1, -x^0)], \quad (65)$$

$$U_\eta^{-1} Q^\xi U_\eta = \mathcal{W}^\xi = \frac{1}{2\sqrt{\pi}} [\bar{\alpha}Q^\xi - \bar{\beta}Q^{-\xi}], \quad \text{where for: } U_\eta = \exp F_\eta, \quad (66)$$

$$F_\eta = 2i\eta \int_{-\infty}^{\infty} dy^1 \varphi^\xi(y^\xi) \partial_0 \varphi^{-\xi}(-y^\xi) = 2i\eta \int_{-\infty}^{\infty} dy^1 \omega^\xi(y^\xi) \partial_0 \omega^{-\xi}(-y^\xi), \quad (67)$$

which in fact does not depend on ξ and x^0 at all, reading as:

$$F_\eta = \frac{i\eta}{2} \int_{-\infty}^{\infty} dy^1 \left\{ \phi(y^1, -x^0) \partial_0 \phi(y^1, x^0) - \phi(y^1, x^0) \partial_0 \phi(y^1, -x^0) \right\} = \mathbf{F}_\eta[c] \equiv \quad (68)$$

$$\equiv \lim_{\eta(k^1) \rightarrow \eta} \int_{-\infty}^{\infty} \frac{dk^1 \theta(k^1)}{2\pi 2k^0} \eta(k^1) \left(c(k^1) c(-k^1) - c^\dagger(-k^1) c^\dagger(k^1) \right), \quad \text{and with (44), (48), (50), gives:} \quad (69)$$

$$[\varphi^{\xi(\pm)}(s), F_\eta] = \eta \varphi^{-\xi(\mp)}(-s), \quad [Q^{\xi(\pm)}(\hat{x}^0), F_\eta] = -\eta Q^{-\xi(\mp)}(-\hat{x}^0), \quad \text{where: } \hat{x}^0 \Rightarrow 0, \quad (70)$$

to have one and the same result (66) by making use of this commutator or relations (32), (38).

In what follows, we use transformation formulas evident for any operators of the form $A = A^{(+)} + A^{(-)}$ and $B = B^{(+)} + B^{(-)}$, whose transformation $U_\eta^{-1} A U_\eta$ for some U_η also depends linearly on the annihilation (+) and creation (-) operators (44), so that all their mutual commutators are c-numbers [9]:

$$U_\eta^{-1} A U_\eta = (U_\eta^{-1} A U_\eta)^{(+)} + (U_\eta^{-1} A U_\eta)^{(-)}, \quad A^{(+)}|0\rangle = (U_\eta^{-1} A U_\eta)^{(+)}|0\rangle = 0, \quad \text{and since:} \quad (71)$$

$$\mathcal{N}_A \{e^A\} \Rightarrow \exp(A^{(-)}) \exp(A^{(+)}) = e^A \exp\left(-\frac{1}{2} [A^{(+)}, A^{(-)}]\right) \equiv \frac{e^A}{\langle 0|e^A|0\rangle}, \quad \text{then:} \quad (72)$$

$$\mathcal{N}_A \{e^A\} \mathcal{N}_B \{e^B\} = \mathcal{N} \{e^{A+B}\} \exp([A^{(+)}, B^{(-)}]) = \mathcal{N}_B \{e^B\} \mathcal{N}_A \{e^A\} \exp([A, B]), \quad (73)$$

$$[\mathcal{N}_A \{e^A\}, \mathcal{N}_B \{e^B\}]_{\mp} = [\exp([A^{(+)}, B^{(-)}]) \mp \exp([B^{(+)}, A^{(-)}])] \mathcal{N} \{e^{A+B}\}, \quad (74)$$

$$B \mathcal{N}_A \{e^A\} = \mathcal{N} \left\{ \left(B + [B^{(+)}, A^{(-)}] \right) e^A \right\}, \quad [B, \mathcal{N}_A \{e^A\}] = [B, A] \mathcal{N}_A \{e^A\}, \quad \text{whence:} \quad (75)$$

$$U_\eta^{-1} \mathcal{N}_A \{e^A\} U_\eta = \mathcal{N}_A \left\{ \exp(U_\eta^{-1} A U_\eta) \right\} \exp\left(\frac{w(\eta)}{2}\right), \quad \text{where:} \quad (76)$$

$$w(\eta) = \left[(U_\eta^{-1} A U_\eta)^{(+)}, (U_\eta^{-1} A U_\eta)^{(-)} \right] - [A^{(+)}, A^{(-)}] = \langle 0| [U_\eta^{-1}, A A] U_\eta |0\rangle, \quad - \text{ is a c-number.} \quad (77)$$

To avoid meaningless infinities the field $\varphi^{-\xi}(s)$ is smoothed by function $f(s)$ [7, 33], which at the end tends to delta-function: $f(s) \rightarrow 2\sqrt{\pi} \delta(s - x^{-\xi})$, so that:

$$\chi_\xi(x^{-\xi}) \mapsto \mathcal{N}_\xi \left\{ \exp A_\xi[f] \right\} u_\xi, \quad A_\xi[f] = -i \left[\int ds \varphi^{-\xi}(s) f(s) + \xi \frac{\sqrt{\pi}}{2} Q^\xi \right] = -i \left[\varphi^{-\xi}[f] + \xi \frac{\sqrt{\pi}}{2} Q^\xi \right], \quad (78)$$

$$\text{leading to convolution: } [\varphi^{\xi(+)}[f], \varphi^{\xi(-)}[g]] = \frac{1}{i} \int ds f(s) \int d\tau D^{(-)}(s - \tau) g(\tau) \equiv \frac{1}{i} (f * D^{(-)} * g), \quad (79)$$

which appears in $w(\eta)$, as: $\frac{1}{i} \left(f * D^{(-)} * f \right) \longrightarrow \frac{4\pi}{i} D^{(-)}(0) = \int_{\mu}^{\infty} \frac{d\lambda}{\lambda} e^{-\lambda/\Lambda} = \ln \frac{\Lambda}{\bar{\mu}}, \quad \bar{\mu} = \mu e^{C_{\Xi}}. \quad (80)$

Here C_{Ξ} is Euler-Mascheroni constant and the ultraviolet cut-off Λ in (80) is introduced by means of representations (317), (329), (336), [13]–[24], thus appearing in the transformed field as:

$$U_{\eta}^{-1} \chi_{\xi} \left(x^{-\xi} \right) U_{\eta} = \lambda_{\xi} \left(x^{-\xi} \right) = \mathcal{N}_{\varphi} \left\{ \exp \left(-i2\sqrt{\pi} \left[\omega^{-\xi} \left(x^{-\xi} \right) + \frac{\xi}{4} \mathcal{W}^{\xi} \right] \right) \right\} v_{\xi}, \quad (81)$$

$$v_{\xi} = \exp \left\{ -a_0 \frac{\bar{\beta}^2}{16} \right\} \left(\frac{\bar{\mu}}{\Lambda} \right)^{\bar{\beta}^2/4\pi} u_{\xi} = \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} \left(\frac{\bar{\mu}}{\Lambda} \right)^{\bar{\beta}^2/4\pi} \exp \left\{ -a_0 \frac{\pi}{4} \left(\frac{1}{2} + \frac{\bar{\beta}^2}{4\pi} \right) \right\} e^{i\varpi - i\xi\Theta/4}. \quad (82)$$

For the corresponding current $\hat{J}_{(\lambda)}^{\mu}(x)$, defined by Eqs. (20)–(22), or by the Johnson definition [2, 3, 7], but with the same renormalization constant $Z_{(\lambda)}(a)$, we find the previous bosonization rules (25):

$$\hat{J}_{(\lambda)}^{\mu}(x) = \frac{1}{\sqrt{\pi}} \partial^{\mu} \omega(x) = -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_{\nu} \Omega(x), \quad \text{for: } Z_{(\lambda)}(a) = (\Lambda^2 |a^2|)^{-\bar{\beta}^2/4\pi}, \quad (83)$$

onto the new (pseudo) scalar fields $\omega(x)$, $\Omega(x)$, $\omega^{\xi}(x^{\xi})$, \mathcal{W}^{ξ} , (64)–(66), obeying the same respective commutation relations (34)–(38):

$$[\omega(x), \omega(y)] = [\Omega(x), \Omega(y)] = -\frac{i}{2} \varepsilon(x^0 - y^0) \theta \left((x - y)^2 \right), \quad (84)$$

$$\left[\omega^{\xi}(s), \omega^{\xi'}(\tau) \right] = -\frac{i}{4} \varepsilon(s - \tau) \delta_{\xi, \xi'}, \quad \left[\omega^{\xi}(s), \mathcal{W}^{\xi'} \right] = \frac{i}{2} \delta_{\xi, \xi'}. \quad (85)$$

Substituting the normal form (81) into the solution (61), we immediately obtain the normal exponential of the DM for Thirring field in the form⁵ of Ref. [33, 34]:

$$\Psi_{\xi}(x) = \mathcal{N}_{\varphi} \left\{ \exp \left[-i\bar{\alpha}\varphi^{-\xi} \left(x^{-\xi} \right) - i\frac{\beta g}{2\pi} \varphi^{\xi} \left(x^{\xi} \right) - i \left(\bar{\beta} - \frac{\beta g}{2\pi} \right) \varphi^{\xi} \left(-x^{-\xi} \right) - i\bar{\alpha}\frac{\xi}{4} Q^{\xi} + i\bar{\beta}\frac{\xi}{4} Q^{-\xi} \right] \right\} v_{\xi}, \quad (86)$$

$$\text{recasted into: } \Psi_{\xi}(x) = \mathcal{N}_{\varphi} \left\{ \exp \left[-i\bar{\alpha}\varphi^{-\xi} \left(x^{-\xi} \right) - i\bar{\beta}\varphi^{\xi} \left(x^{\xi} \right) - i\bar{\alpha}\frac{\xi}{4} Q^{\xi} + i\bar{\beta}\frac{\xi}{4} Q^{-\xi} \right] \right\} v_{\xi}, \quad (87)$$

$$\text{or: } \Psi_{\xi}(x) = \mathcal{N}_{\varphi} \left\{ \exp \left(-i2\sqrt{\pi} \left[\varrho^{-\xi}(x) + \frac{\xi}{4} \mathcal{W}^{\xi} \right] \right) \right\} v_{\xi}, \quad \varrho^{-\xi}(x) = \frac{1}{2\sqrt{\pi}} \left[\bar{\alpha}\varphi^{-\xi} \left(x^{-\xi} \right) + \bar{\beta}\varphi^{\xi} \left(x^{\xi} \right) \right], \quad (88)$$

by imposing the conditions onto the parameters, that are necessary to have a correct Lorentz -transformation properties corresponding to the spin 1/2 and correct CAR (8)–(10), respectively:

$$\bar{\alpha}^2 - \bar{\beta}^2 = 4\pi, \quad \bar{\beta} - \frac{\beta g}{2\pi} = 0. \quad (89)$$

⁵With taking again into account the difference in $-\sqrt{2}$ of the field's and charge's normalization, and the losted coefficient 1/4 instead 1/2 before the charges in [33] (11.129), (11.130).

Straightforward calculation of the vector current operators (20)–(22) for this solution (87), (88), (82) by means of Eqs. (25), (27), (38)–(42) and (89), under the conditions:

$$\bar{\alpha} \equiv \left(\frac{2\pi}{\kappa} + \frac{\kappa}{2} \right), \quad \bar{\beta} \equiv \left(\frac{2\pi}{\kappa} - \frac{\kappa}{2} \right), \quad \text{or: } e^\eta \equiv \frac{2\sqrt{\pi}}{\kappa}, \quad (90)$$

reproduces exactly the bosonization and linearization relations (23), (24), (25) as following:

$$\hat{J}_{(\Psi)}^\mu(x) \stackrel{w}{=} -\frac{\kappa}{2\pi} \epsilon^{\mu\nu} \partial_\nu \phi(x) = \frac{\beta}{2\sqrt{\pi}} \hat{J}_{(\chi)}^\mu(x), \quad \text{for: } \kappa = \beta, \quad (91)$$

$$Z_{(\chi)}(a) = 1, \quad Z_{(\Psi)}(a) = (-\Lambda^2 a^2)^{-\bar{\beta}^2/4\pi} \propto Z_{(\lambda)}(a), \quad \text{whence:} \quad (92)$$

$$\bar{\alpha} = \left(\frac{2\pi}{\beta} + \frac{\beta}{2} \right), \quad \bar{\beta} = \left(\frac{2\pi}{\beta} - \frac{\beta}{2} \right), \quad \text{or: } e^\eta = \frac{2\sqrt{\pi}}{\beta} = \sqrt{1 + \frac{g}{\pi}}, \quad (93)$$

demonstrating self-consistency of the above calculations. The last equality of Eq. (93) is easily recognized as the well-known Coleman identity [8]. The weak sense of bosonization rules (91), unlike (25), is directly manifested by the difference of renormalization constants $Z_{(\Psi)}(a)$ and $Z_{(\chi)}(a)$ defined by Eq. (92) for the various fields $\Psi(x)$ and $\chi(x)$ respectively.

Further we also refer to the solution (87), (88), (82), as Oksak solution $\Psi^{Ok}(x)$, [33, 34]. It is worth to note it contains all Klein factors also under the normal form, as is demanded for DM. By virtue of (74) the value of $Z_{(\Psi)}(x-y)$ is fixed also as the wave function renormalization constant, which defines dynamical dimension $d_{(\Psi)}$ of the Thirring field [5, 6, 23] by replacing the CAR (8) to the following relation:

$$\left\{ \Psi_\xi(x), \Psi_{\xi'}^\dagger(y) \right\} \Big|_{x^0=y^0} = Z_{(\Psi)}(x-y) \Big|_{x^0=y^0} \delta_{\xi,\xi'} \delta(x^1-y^1), \quad (94)$$

$$Z_{(\Psi)}(x-y) = \frac{2\pi}{\bar{\mu}} \left(-\bar{\mu}^2(x-y)^2 \right)^{-\bar{\beta}^2/4\pi} \exp \left\{ a_0 \frac{\pi}{2} \left(\frac{1}{2} + \frac{\bar{\beta}^2}{4\pi} \right) \right\} (v_\xi^* v_\xi), \quad \text{or:} \quad (95)$$

$$Z_{(\Psi)}(x-y) \Big|_{x^0=y^0} = \left[-\Lambda^2(x-y)^2 \right]^{-\bar{\beta}^2/4\pi} \Big|_{x^0=y^0} = \left[\Lambda^2(x^1-y^1)^2 \right]^{-\bar{\beta}^2/4\pi}, \quad d_{(\Psi)} = \frac{1}{2} + \frac{\bar{\beta}^2}{4\pi}, \quad (96)$$

whence: $Z_{(\Psi)}|_{x^0=y^0} \rightarrow 1$, for: $x^1 - y^1 \simeq 1/\Lambda$. For the intermediate field $\lambda_\xi(x^{-\xi})$ (81), (82) one finds the same renormalization constant $Z_{(\lambda)}|_{x^0=y^0} = Z_{(\Psi)}|_{x^0=y^0}$ (96).

The following comments are in order.

1. Our way to obtain the Oksak solution for the Thirring field by means of direct integration of exactly linearized HEq with initial condition at $t = x^0 = 0$ gives it as a multiplicatively renormalizable operator, whose current's (92) and field's (96) renormalization constants depend only on ultraviolet cut-off Λ as it should. All the infrared regularization parameters $\bar{\mu}$ and a_0 are canceled automatically. Moreover, the parameter⁶ a_0 fully eliminates from the Oksak-Thirring field (87), (82), as well as from the free one (56) by means of the one and the same redefinition of parameter $\bar{\mu}$ as:

$$\bar{\mu} \longmapsto \bar{\bar{\mu}} \exp \left\{ a_0 \frac{\pi}{4} \right\}. \quad (97)$$

⁶which is zero in original Oksak solution [33, 34] (see Appendix C).

2. The general form of our solution (61), (62) is very close to the Klaiber operator solution [5], but is not the same. The main difference manifests in the inequivalent representation for the “intermediate” free Dirac field arisen as “initial” HF: $\lambda_\xi(x^{-\xi}) = \Psi_\xi(x^1 - \xi x^0, 0)$, and induced by inequivalent representation of the (pseudo) scalar field. The appearance of ultraviolet cut-off Λ in (82) is directly related to well known non-existence of HF at fixed point of time [33, 40], also leading to necessity to deal with inequivalent representations for describing the time evolution of non-trivial interaction [33, 41]. The fields and the vacuum state $|\widehat{0}\rangle$ for this inequivalent representation are defined by the relations (64)–(70) as:

$$|\widehat{0}\rangle = U_\eta^{-1}|0\rangle, \quad d(k^1)|\widehat{0}\rangle = 0, \quad d(k^1) = U_\eta^{-1}c(k^1)U_\eta = \frac{1}{2\sqrt{\pi}} [\bar{\alpha}c(k^1) - \bar{\beta}c^\dagger(-k^1)], \quad (98)$$

$$d^\dagger(k^1) = U_\eta^{-1}c^\dagger(k^1)U_\eta = \frac{1}{2\sqrt{\pi}} [\bar{\alpha}c^\dagger(k^1) - \bar{\beta}c(-k^1)], \quad [d(k^1), d^\dagger(q^1)] = 2\pi 2k^0\delta(k^1 - q^1), \quad (99)$$

$$\widehat{\omega}^{\xi(+)}(s) = -\frac{\xi}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \theta(-\xi k^1) d(k^1) e^{-ik^0 s} = \frac{1}{2\sqrt{\pi}} [\bar{\alpha}\varphi^{\xi(+)}(s) + \bar{\beta}\varphi^{-\xi(-)}(-s)], \quad k^0 = |k^1|, \quad (100)$$

$$\widehat{\omega}^{\xi(-)}(s) = [\widehat{\omega}^{\xi(+)}(s)]^\dagger, \quad \omega^\xi(s) = \widehat{\omega}^{\xi(+)}(s) + \widehat{\omega}^{\xi(-)}(s) = \omega^{\xi(+)}(s) + \omega^{\xi(-)}(s), \quad (101)$$

$$\text{while: } \omega^{\xi(+)}(s) = \frac{1}{2\sqrt{\pi}} [\bar{\alpha}\varphi^{\xi(+)}(s) + \bar{\beta}\varphi^{-\xi(+)}(-s)], \quad \omega^{\xi(-)}(s) = [\omega^{\xi(+)}(s)]^\dagger, \quad (102)$$

$$\widehat{\lambda}_\xi(x^{-\xi}) = \mathcal{N}_\omega \left\{ \exp \left(-i2\sqrt{\pi} \left[\omega^{-\xi}(x^{-\xi}) + \frac{\xi}{4} \mathcal{W}^\xi \right] \right) \right\} u_\xi \leftarrow \chi_\xi(x^{-\xi}) \left[\text{for: } \varphi^\xi \mapsto \omega^\xi, Q^\xi \mapsto \mathcal{W}^\xi \right]. \quad (103)$$

The latter replacement of (103) in (25)–(56) (see also (84)–(85)) fully determines this representation as “almost” equivalent to the previous one up to replacing (pseudo) scalar field’s representation space (98)–(101) and corresponding definition of the free fermionic fields (55) \mapsto (103) \mapsto (81), (82), and their currents definitions (25), (83): the current’s renormalization constant (22), $Z_{(\widehat{\lambda})}(a) = Z_{(\chi)}(a) = 1$, due to the normal-ordering with respect to different spaces, but as well as the current’s (83), as well as the field’s renormalization constant (96) for the one and the same field $\lambda(x)$, it becomes representation-dependent. The unitarily inequivalent nature (82) of free Dirac intermediate field $\lambda_\xi(x^{-\xi})$ manifests itself namely due to its normal-ordering \mathcal{N}_φ in (81) with respect to the initial vacuum $|0\rangle$ of Eq. (44), instead of \mathcal{N}_ω (103) with respect to $|\widehat{0}\rangle$. Other details of this unitarily inequivalent transformation are pointed in Appendix A.

2.5 Schwinger’s terms, other solutions, and superselection rules

By making use of (25), (75), (345), (346), (34), for the Johnson non-equal time and equal time commutators [2]–[8] of Heisenberg fields (87), (88) and their currents (91), understanding in a weak sense, one has:

$$[\widehat{J}_{(\Psi)}^\mu(x), \Psi(y)] \stackrel{w}{=} -(\underline{a}_{(\Psi)}g^{\mu\nu} - \bar{a}_{(\Psi)}\gamma^5\epsilon^{\mu\nu}) \frac{\partial}{\partial x^\nu} D_0(x-y)\Psi(y), \quad (104)$$

$$[\widehat{J}_{(\Psi)}^0(x), \Psi(y)] \Big|_{x^0=y^0} \stackrel{w}{=} -\underline{a}_{(\Psi)}\Psi(y)\delta(x^1-y^1), \quad (105)$$

$$[\widehat{J}_{(\Psi)}^1(x), \Psi(y)] \Big|_{x^0=y^0} \stackrel{w}{=} -\bar{a}_{(\Psi)}\gamma^5\Psi(y)\delta(x^1-y^1), \quad (106)$$

$$[\widehat{J}_{(\Psi)}^0(x), \widehat{J}_{(\Psi)}^1(y)] \Big|_{x^0=y^0} \stackrel{w}{=} -ic_{(\Psi)}\partial_{x^1}\delta(x^1-y^1), \quad (107)$$

and upon the above accepted definitions (20)–(22), (91), and relations (92), (93), we obtain:

$$\underline{a}_{(\Psi)} = 1, \quad \bar{a}_{(\Psi)} = \frac{\beta^2}{4\pi}, \quad c_{(\Psi)} = \frac{\beta^2}{4\pi^2}, \quad \text{and find,} \quad (108)$$

$$\text{that: } \underline{a}_{(\Psi)} \bar{a}_{(\Psi)} = \pi c_{(\Psi)}, \quad \underline{a}_{(\Psi)} - \bar{a}_{(\Psi)} = g c_{(\Psi)}, \quad (109)$$

in agreement with [3]–[8]. On the other hand, in accordance with [9], [45], [46], the algebra of the Heisenberg operator of the conserved fermionic (vector) charge and the Thirring field, by virtue of (105), (108), coincides with the algebra of the conserved fermionic (vector) charge $O_{(\chi)}/\sqrt{\pi}$ and the free trial physical field $\chi(x)$ defined by Eqs. (25), (31), and/or Eqs. (366)–(370), as:

$$Q_{(\Psi)} = \frac{O_{(\Psi)}}{\sqrt{\pi}} = \int_{-\infty}^{\infty} dx^1 \hat{J}_{(\Psi)}^0(x^1, x^0) \stackrel{w}{=} \frac{\beta}{2\pi} O_{(\chi)}, \quad Q_{(\chi)} = \frac{O_{(\chi)}}{\sqrt{\pi}} \equiv \frac{O}{\sqrt{\pi}} = \int_{-\infty}^{\infty} dx^1 \hat{J}_{(\chi)}^0(x^1, x^0), \quad (110)$$

$$\text{whence: } [Q_{(\Psi)}, \Psi(y)] = -\Psi(y), \quad [Q_{(\chi)}, \chi(y)] = -\chi(y), \quad \text{for } \underline{a}_{(\Psi)} = \underline{a}_{(\chi)} = 1. \quad (111)$$

The pseudoscalar (pseudovector) charges $O_{5(\Psi)}$ and $O_{5(\chi)} \equiv O_5$ of Eqs. (25), (31) are related analogously. But because of Eq. (106), $\bar{a}_{(\Psi)} \neq \bar{a}_{(\chi)} = 1$, and their algebras with corresponding fields are different:

$$Q_{5(\Psi)} = \frac{O_{5(\Psi)}}{\sqrt{\pi}} = \int_{-\infty}^{\infty} dx^1 \hat{J}_{(\Psi)}^1(x^1, x^0) \stackrel{w}{=} \frac{\beta}{2\pi} O_{5(\chi)}, \quad Q_{5(\chi)} = \frac{O_{5(\chi)}}{\sqrt{\pi}} = \int_{-\infty}^{\infty} dx^1 \hat{J}_{(\chi)}^1(x^1, x^0), \quad (112)$$

$$\text{whence: } [Q_{5(\Psi)}, \Psi(y)] = -\bar{a}_{(\Psi)} \gamma^5 \Psi(y), \quad [Q_{5(\chi)}, \chi(y)] = -\gamma^5 \chi(y). \quad (113)$$

Note, the use of values (105), (106) for calculation of the commutator in Eq. (13) also violates the equations of motion (13), (14), as well as the above-mentioned attempt to use the commutator (107) in equation (18), what may be compared with [12].

Thus, we come to conclusions, that Thirring model [1]–[6], as well as the Federbush one [45], is exactly solvable due to intrinsic hidden exact linearizability of its HEq, and that operator bosonization rules make sense only among the free fields operators (25) with unambiguously defined procedure of normal ordering. For the Heisenberg currents these rules are applicable only in a weak sense (91).

The natural manifestation of inequivalent representations (55), (81) and (87) of free and Heisenberg fermionic field and their currents are also the various values of Schwinger's terms (107), (108) and dynamical dimension (96):

$$c_{(\chi)} = c_{(\lambda)} = \frac{1}{\pi}, \quad c_{(\Psi)} = \frac{1}{\pi + g}, \quad d_{(\chi)} = \frac{1}{2}, \quad d_{(\Psi)} - \frac{1}{2} = \frac{\bar{\beta}^2}{4\pi} = \frac{g^2}{4\pi} c_{(\Psi)} = \frac{g^2}{4\pi} \frac{1}{(\pi + g)}, \quad (114)$$

in agreement with [5, 6, 11, 23]. Similarly to the solution [45] of Federbush model, the linear homogeneous HEq (58) does not define the normalization (96) of HF (87), (93), which, as well as for the free fields $\chi(x)$, $\lambda(x)$ is fixed [21] only by the anticommutation relations (8) \mapsto (94).

We would like to emphasize again, that unlike [35]–[37] the bosonization procedure of Refs. [8]–[33] is considered here as a particular case of dynamical mapping onto the Schrödinger physical field [43]–[45] defined at $t = 0$. From this view point the results of Refs. [25] and [13] may be considered as DM

of Thirring field onto the free massive scalar field $\phi_m(x)$ and onto the free massive Dirac field $\psi_M(x)$ respectively. The general form of solution (61) should give a possibility to describe all phases of the theory under consideration. We show such example for finite temperature in the next section.

Now we wish to connect the Oksak solution with another known solutions of Thirring model [9, 27]. To this end we use the formally unitary transformation of conformal shift for scalar fields from Ref. [34], generalized by the following way. By making use of the relations (41), (42), (76), (77) and keeping in mind the Eqs. (93), we consider the family of solutions marked by arbitrary real parameter σ :

$$K_\sigma = \exp X_\sigma, \quad X_\sigma = i\sigma \frac{\bar{\xi}}{4} (Q^{-\bar{\xi}} Q^{-\xi} - Q^{\bar{\xi}} Q^{\xi}) = i\frac{\sigma}{4} O O_5, \quad \bar{\xi}, \xi = \pm \quad (115)$$

$$\Psi(x, \sigma) = K_\sigma^{-1} \Psi^{Ok}(x) K_\sigma, \quad \Psi_\xi(x, \sigma) = K_\sigma^{-1} \Psi_\xi^{Ok}(x) K_\sigma = \mathcal{N}_\varphi \left\{ e^{R_\xi(x, \sigma)} \right\} v_\xi(\sigma), \quad (116)$$

$$R_\xi(x, \sigma) = -i \left[\bar{\alpha} \varphi^{-\xi} (x^{-\xi}) + \bar{\beta} \varphi^\xi (x^\xi) + \frac{\xi}{4} (\bar{\alpha} + \sigma \bar{\beta}) Q^\xi - \frac{\xi}{4} (\bar{\beta} + \sigma \bar{\alpha}) Q^{-\xi} \right], \quad (117)$$

$$v_\xi(\sigma) = \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} \left(\frac{\bar{\mu}}{\Lambda} \right)^{\bar{\beta}^2/4\pi} \exp \left\{ -a_0 \frac{\pi}{8} [(\sigma^2 + 1) \cosh 2\eta + 2\sigma \sinh 2\eta] \right\} e^{i\varpi - i\xi\Theta/4}. \quad (118)$$

This solution obeys the same CAR (9), (10), (94), and the bosonization rule (91) with the same renormalization constant $Z_{(\Psi)}(a)$ (92), (96), for arbitrary σ , and the parameter a_0 may be again adsorbed to the parameter $\bar{\mu}$ by the rescaling substitution, which unlike the Oksak and free cases, now depends on Thirring coupling constant:

$$\bar{\mu} \mapsto \bar{\mu} \exp \left\{ a_0 \frac{\pi}{4} (\sigma^2 + 1 + 2\sigma \tanh 2\eta) \right\}, \quad (119)$$

By using Eqs. (28), (31), it is a simple matter to check that $\sigma = \pm 1$ gives the two types of Mandelstam solution [9], while $\sigma = -\coth 2\eta$ corresponds to normal form for solution of Morchio et al. [27]. This again demonstrates the advantages of normal ordered form of HF demanded by DM:

$$\Psi_\xi(x, 1) = \mathcal{N}_\varphi \left\{ e^{R_\xi(x, 1)} \right\} \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} \left(\frac{\bar{\mu}}{\Lambda} \right)^{\bar{\beta}^2/4\pi} \exp \left\{ -a_0 \frac{\pi^2}{\beta^2} \right\} e^{i\varpi - i\xi\Theta/4}, \quad (120)$$

$$R_\xi(x, 1) = -i \left[\xi \frac{\beta}{2} \phi(x^1, x^0) - \frac{2\pi}{\beta} \int_{-\infty}^{x^1} dy^1 \partial_0 \phi(y^1, x^0) \right], \quad \sigma = 1; \quad (121)$$

$$\Psi_\xi(x, -1) = \mathcal{N}_\varphi \left\{ e^{R_\xi(x, -1)} \right\} \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} \left(\frac{\bar{\mu}}{\Lambda} \right)^{\bar{\beta}^2/4\pi} \exp \left\{ -a_0 \frac{\beta^2}{16} \right\} e^{i\varpi - i\xi\Theta/4}, \quad (122)$$

$$R_\xi(x, -1) = -i \left[\frac{2\pi}{\beta} \varphi(x^1, x^0) + \xi \frac{\beta}{2} \int_{x^1}^{\infty} dy^1 \partial_0 \varphi(y^1, x^0) \right], \quad \sigma = -1; \quad (123)$$

$$\Psi_\xi(x, -\coth 2\eta) = \mathcal{N}_\varphi \left\{ e^{R_\xi(x, -\coth 2\eta)} \right\} \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} \left(\frac{\bar{\mu}}{\Lambda} \right)^{\bar{\beta}^2/4\pi} \exp \left\{ -a_0 \frac{\pi}{8} \frac{\cosh 2\eta}{\sinh^2 2\eta} \right\} e^{i\varpi - i\xi\Theta/4}, \quad (124)$$

$$R_\xi(x, -\coth 2\eta) = -i \left[\bar{\alpha} \varphi^{-\xi} (x^{-\xi}) + \bar{\beta} \varphi^\xi (x^\xi) + \xi \frac{\pi}{2} \left(\frac{Q^\xi}{\bar{\alpha}} + \frac{Q^{-\xi}}{\bar{\beta}} \right) \right], \quad \sigma = -\coth 2\eta. \quad (125)$$

We would like to point out that $\sigma = 1$ corresponds to DM (120), (121) onto the pseudoscalar field $\phi(x)$, while $\sigma = -1$ gives another form of solution (122), (123) with bosonization onto the scalar field $\varphi(x)$, and that, unlike (124), (125), the original solution of Morchio et al. [27] has $a_0 = 0$, and contains all Klein factors outside the normal form, thus its renormalization constant remains to be unknown. Here and below we use for brevity the mixed notations from identities (89), (93), and the following relations:

$$\begin{aligned}\bar{\alpha} + \bar{\beta} &= \frac{4\pi}{\beta} = 2\sqrt{\pi}e^\eta, \quad \bar{\alpha} - \bar{\beta} = \beta = 2\sqrt{\pi}e^{-\eta}, \quad \bar{\alpha}^2 - \bar{\beta}^2 = 4\pi, \\ \frac{\bar{\alpha}\bar{\beta}}{4\pi} &= \frac{1}{4} \left(\frac{4\pi}{\beta^2} - \frac{\beta^2}{4\pi} \right) = \frac{\sinh 2\eta}{2}, \quad \frac{1}{2} + \frac{\bar{\beta}^2}{4\pi} = \frac{\bar{\alpha}^2 + \bar{\beta}^2}{8\pi} = \frac{1}{4} \left(\frac{4\pi}{\beta^2} + \frac{\beta^2}{4\pi} \right) = \frac{\cosh 2\eta}{2} = d_{(\Psi)}.\end{aligned}\quad (126)$$

Now let us turn to the VEV [7] of the strings of these fields (116). Following [33] it is enough and convenient to consider only the product:

$$\left\langle 0 \left| \prod_{i=1}^p \Psi_{\xi_i}^{(l_i)}(x_i, \sigma) \right| 0 \right\rangle = ? \quad , \quad \text{where: } l = +1, \text{ for } \Psi, \quad \text{and } l = -1, \text{ for } \Psi^\dagger. \quad (127)$$

By virtue of generalized formula (73), which for any $R_{\xi_i}^{(l_i)}(x_i, \sigma) \equiv l_i R_{\xi_i}(x_i, \sigma) \mapsto \mathcal{R}_i = \mathcal{R}_i^{(+)} + \mathcal{R}_i^{(-)}$, may be easily checked by induction:

$$\mathcal{N}\{\exp[\mathcal{R}_1]\} \cdots \mathcal{N}\{\exp[\mathcal{R}_p]\} \equiv \prod_{i=1}^p \mathcal{N}\{\exp[\mathcal{R}_i]\} = \exp\left\{\sum_{i < k}^p [\mathcal{R}_i^{(+)}, \mathcal{R}_k^{(-)}]\right\} \mathcal{N}\left\{\exp\left(\sum_{j=1}^p \mathcal{R}_j\right)\right\}, \quad (128)$$

and obvious relations for any numbers γ_i , $i = 1 \div p$:

$$\sum_{i < k}^p \gamma_i \gamma_k = \frac{1}{2} \sum_{i \neq k}^p \gamma_i \gamma_k = \frac{1}{2} \left(\sum_{i=1}^p \gamma_i \right)^2 - \frac{1}{2} \sum_{i=1}^p \gamma_i^2, \quad (129)$$

with the help of formulas (326)–(330) from Appendix B, one can obtain for the solutions (116):

$$\begin{aligned}\left\langle 0 \left| \prod_{i=1}^p \Psi_{\xi_i}^{(l_i)}(x_i, \sigma) \right| 0 \right\rangle &= \left\langle 0 \left| \mathcal{N}_\varphi \left\{ \exp \left(\sum_{j=1}^p \mathcal{R}_j \right) \right\} \right| 0 \right\rangle \exp \left\{ i\varpi \sum_{i=1}^p l_i - i\frac{\Theta}{4} \sum_{i=1}^p l_i \xi_i \right\} \\ &\cdot \left(\frac{\bar{\mu}}{2\pi} \right)^{\frac{p}{2}} \left(\frac{\bar{\mu}}{\Lambda} \right)^{\frac{p}{4\pi} \bar{\beta}^2} \frac{1}{\bar{\mu}} \left(\bar{\alpha}^2 + \bar{\beta}^2 \right)^{\frac{1}{2 \cdot 8\pi}} \left[\left(\sum_{i=1}^p l_i \right)^2 + \left(\sum_{i=1}^p l_i \xi_i \right)^2 - 2p \right] \frac{1}{\bar{\mu}} \bar{\alpha} \bar{\beta} \left[\left(\sum_{i=1}^p l_i \right)^2 - \left(\sum_{i=1}^p l_i \xi_i \right)^2 \right] \\ &\cdot e^{-pa_0 \frac{\pi}{8} [(\sigma^2 + 1) \cosh 2\eta + 2\sigma \sinh 2\eta]} e^{-\frac{1}{43} (1+\sigma)^2 (\bar{\alpha} + \bar{\beta})^2 a_0 \left[\left(\sum_{i=1}^p l_i \right)^2 - p \right]} e^{-\frac{1}{43} (1-\sigma)^2 (\bar{\alpha} - \bar{\beta})^2 a_0 \left[\left(\sum_{i=1}^p l_i \xi_i \right)^2 - p \right]} \\ &\cdot \prod_{i < k}^p \left[e^{i2\pi^2 (\xi_i - \xi_k)} \left(i \{ x_i^{-\xi_i} - x_k^{-\xi_i} \} + 0 \right)^{[(\bar{\alpha}^2 + \bar{\alpha} \bar{\beta}) + (\bar{\alpha}^2 - \bar{\alpha} \bar{\beta}) \xi_i \xi_k]} \left(i \{ x_i^{\xi_i} - x_k^{\xi_i} \} + 0 \right)^{[(\bar{\beta}^2 + \bar{\alpha} \bar{\beta}) + (\bar{\beta}^2 - \bar{\alpha} \bar{\beta}) \xi_i \xi_k]} \right]^{\frac{1}{8\pi} l_i l_k},\end{aligned}$$

or, after some simplifications⁷:

$$\begin{aligned}
& \left(\Lambda^{\overline{\beta}^2/4\pi\sqrt{2\pi}} \right)^p \left\langle 0 \left| \prod_{i=1}^p \Psi_{\xi_i}^{(l_i)}(x_i, \sigma) \right| 0 \right\rangle = \left\langle 0 \left| \mathcal{N}_\varphi \left\{ \exp \left(\sum_{j=1}^p \mathcal{R}_j \right) \right\} \right| 0 \right\rangle \exp \left\{ i\varpi \sum_{i=1}^p l_i - i\frac{\Theta}{4} \sum_{i=1}^p l_i \xi_i \right\} \\
& \cdot \left(\overline{\mu} \exp \left\{ -a_0 \frac{\pi}{4} (1 + \sigma)^2 \right\} \right)^{\left(\sum_{i=1}^p l_i \right)^2 (\pi/\beta^2)} \left(\overline{\mu} \exp \left\{ -a_0 \frac{\pi}{4} (1 - \sigma)^2 \right\} \right)^{\left(\sum_{i=1}^p l_i \xi_i \right)^2 (\beta^2/16\pi)} \\
& \cdot \prod_{i < k}^p \left\{ e^{i\pi(\xi_i - \xi_k)} \left[\frac{i(x_i^- - x_k^-) + 0}{i(x_i^+ - x_k^+) + 0} \right]^{\xi_i + \xi_k} \left[i0\varepsilon(x_i^0 - x_k^0) - (x_i - x_k)^2 \right]^{(4\pi/\beta^2) + \xi_i \xi_k (\beta^2/4\pi)} \right\}^{l_i l_k / 4}. \quad (130)
\end{aligned}$$

The first line and second line of this expression provide at least five independent sources of the superselection rules [7, 33], which usually are associated only with conservation of scalar field's (vector current's) charge O (110), and pseudoscalar field's (pseudovector current's) charge O_5 (112), respectively:

$$\sum_{i=1}^p l_i = 0, \quad (131)$$

$$\sum_{i=1}^p l_i \xi_i = 0. \quad (132)$$

The first one defined by Oksak and Morchio et al., due to above mentioned charge conservation, originates from the VEV of normal exponential in r.h.s. of the first line, taken instead $|0\rangle$ for the vacuum state $|\hat{v}\rangle$ averaged with respect to the field-translation gauge group (52), leading to [27, 33, 34]:

$$\left\langle \hat{v} \left| \mathcal{N}_\varphi \left\{ \exp \left(\sum_{j=1}^p \mathcal{R}_j \right) \right\} \right| \hat{v} \right\rangle \Rightarrow \delta_{\sum_{i=1}^p l_i, 0} \delta_{\sum_{i=1}^p l_i \xi_i, 0}. \quad (133)$$

The VEV of this normal form for the usual non-degenerate vacuum state $|0\rangle$ is equal to 1 identically [10, 13, 21]. Nevertheless, these rules arise from the second line at the limit $\overline{\mu} \rightarrow 0$ as the natural conditions of nonzero result [13]. We can suggest now three additional sources of these rules: the third one is the ϖ - and Θ - independence condition for the VEV (127), (130), the fourth one follows from its above mentioned independence on the parameter a_0 , and the fifth one follows from its independence on σ , if the transformation (115) leaves the vacuum invariant. Obviously independence on the a_0 automatically means the independence on σ and vice versa.

The independence on the initial random values of overall and relative phases has purely fermionic nature and does not reduce to the (pseudo) scalar field-translation gauge symmetry (52), which can only shift their arbitrary initial values. The fate of a_0 - independence of the VEV is more delicate, because for Oksak solution, $\sigma = 0$, it may be again eliminated from the second line of (130) by redefinition (97) of the $\overline{\mu}$. But that is not the case for arbitrary σ . It is worth to note the superselection rules (131), (132) leave necessary for VEV (127) only the ultraviolet renormalization in the l.h.s of the first line of Eq. (130).

⁷Here we put $\xi_i, \xi_k = \pm 1$, for $\xi_i + \xi_k$, and $\xi_i - \xi_k$, and use: $\delta_{\xi_i, \xi_k} = (1 + \xi_i \xi_k)/2$, $\delta_{\xi_i, 1} = (1 + \xi_i)/2$, $\delta_{\xi_i, -1} = (1 - \xi_i)/2$.

Whence, only the third line of (130) survives [33], which is always independent of parameters $\bar{\mu}$, σ , a_0 , ϖ , Θ :

$$\left(\Lambda^{\bar{\beta}^2/4\pi}\right)^p \left\langle 0 \left| \prod_{i=1}^p \Psi_{\xi_i}^{(l_i)}(x_i, \sigma) \right| 0 \right\rangle = \left\langle 0 \left| \prod_{i=1}^p \chi_{\xi_i}^{(l_i)}(x_i) \right| 0 \right\rangle \prod_{i < k}^p \left[i 0 \varepsilon (x_i^0 - x_k^0) - (x_i - x_k)^2 \right]^{\frac{g}{4} \left[\frac{1}{\pi} - \frac{\xi_i \xi_k}{\pi + g} \right] l_i l_k}, \quad (134)$$

$$\begin{aligned} \text{where: } (2\pi)^{p/2} \left\langle 0 \left| \prod_{i=1}^p \chi_{\xi_i}^{(l_i)}(x_i) \right| 0 \right\rangle &\Rightarrow \delta \sum_{i=1}^p l_{i,0} \delta \sum_{i=1}^p l_{i\xi_i,0} \\ &\cdot \prod_{i < k}^p \left\{ e^{i\pi(\xi_i - \xi_k)} \left[i(x_i^- - x_k^-) + 0 \right]^{(1+\xi_i)(1+\xi_k)} \left[i(x_i^+ - x_k^+) + 0 \right]^{(1-\xi_i)(1-\xi_k)} \right\}^{l_i l_k / 4}. \end{aligned} \quad (135)$$

This gives the well known expressions for two-point functions [5, 23] with the dynamical dimension (114).

We see that, unlike Schwinger model [33, 60, 61], as long as we deal with solutions of “phase decoupled” HEqs (14) or (58), that preserve the ϖ and Θ arbitrariness, both the superselection rules (131), (132) with the conservation of both **currents** should be fulfilled independently of chosen phase of the theory, including the phase with spontaneously breaking of chiral symmetry [28, 29]. From this view point the breaking of the rule (132) may be achieved formally only or by introducing the mass term into HEq (13) “by hand” [13], or otherwise, by excluding a_0 via taking the Mandelstam’s solution with $\sigma = 1$ supplemented with fixing of the values $\bar{\mu} \mapsto M$ and Θ [13, 19]. However, as we will see below, the latter way is impossible for the finite temperature case. Moreover, if one of the gauge symmetries remains unbroken: $O|0\rangle \Rightarrow 0$, being connected by transformation (115) all the above solutions refer to the same vacuum state $|0\rangle$.

Keeping in mind the correspondence (80), for $\varepsilon^0 \rightarrow 0$, $\varepsilon^1 \rightarrow 1/\Lambda$, the renormalized operator of scalar “condensate” read:

$$\begin{aligned} \bar{\Psi}(x + \varepsilon) \Psi(x) &= \frac{\bar{\mu}}{\pi} \left(\frac{\bar{\mu}^2}{\Lambda^2} \right)^{\bar{\beta}^2/4\pi} \left(-\bar{\mu}^2 \varepsilon^2 + i 0 \varepsilon^0 \right)^{-\bar{\alpha}\bar{\beta}/4\pi} \exp \left\{ -a_0 \frac{\pi}{4} (1 - \sigma)^2 \frac{\beta^2}{4\pi} \right\} \\ \cdot \mathcal{N}_\phi \left\{ \sin \left(\beta \left[\phi(x) + (1 - \sigma) \frac{O}{4} \right] + \frac{\Theta}{2} \right) \right\}, \quad (\bar{\Psi}(x) \Psi(x))_{ren} &= \lim_{\varepsilon \rightarrow 0} \left(-\Lambda^2 \varepsilon^2 \right)^{\bar{\alpha}\bar{\beta}/4\pi} \bar{\Psi}(x + \varepsilon) \Psi(x), \end{aligned} \quad (136)$$

$$(\bar{\Psi}(x) \Psi(x))_{ren} \Rightarrow \frac{\Lambda}{\pi} \left(\frac{\bar{\mu}}{\Lambda} \exp \left\{ -a_0 \frac{\pi}{4} (1 - \sigma)^2 \right\} \right)^{\beta^2/4\pi} \mathcal{N}_\phi \left\{ \sin \left(\beta \left[\phi(x) + (1 - \sigma) \frac{O}{4} \right] + \frac{\Theta}{2} \right) \right\}, \quad (137)$$

$$\text{where: } \sum_{i=1}^2 l_i = 0, \quad \left(\sum_{i=1}^2 l_i \xi_i \right)^2 = 4, \quad \left(-\bar{\mu}^2 \varepsilon^2 + i 0 \varepsilon^0 \right)^{-\bar{\alpha}\bar{\beta}/4\pi} \Rightarrow \left(\bar{\mu} \varepsilon^1 \right)^{-\bar{\alpha}\bar{\beta}/2\pi} \mapsto \left(\frac{\bar{\mu}}{\Lambda} \right)^{-\bar{\alpha}\bar{\beta}/2\pi}. \quad (138)$$

This operator simplifies for Mandelstam case $\sigma = 1$, and by virtue of (138) its VEV for the vacuum state (133), of course is zero [27, 29], contrary to [21]. For the non degenerate vacuum state $|0\rangle$ it reads:

$$\left\langle 0 \left| (\bar{\Psi}(x) \Psi(x))_{ren} \right| 0 \right\rangle \Rightarrow \frac{\Lambda}{\pi} \left(\frac{\bar{\mu}}{\Lambda} \exp \left\{ -a_0 \frac{\pi}{4} (1 - \sigma)^2 \right\} \right)^{\beta^2/4\pi} \sin \frac{\Theta}{2}. \quad (139)$$

Analogously for pseudoscalar case the same relations (138) take place, leading to:

$$\bar{\Psi}(x + \varepsilon) \gamma^5 \Psi(x) = i \frac{\bar{\mu}}{\pi} \left(\frac{\bar{\mu}^2}{\Lambda^2} \right)^{\bar{\beta}^2/4\pi} \left(-\bar{\mu}^2 \varepsilon^2 + i 0 \varepsilon^0 \right)^{-\bar{\alpha}\bar{\beta}/4\pi} \exp \left\{ -a_0 \frac{\pi}{4} (1 - \sigma)^2 \frac{\beta^2}{4\pi} \right\}$$

$$\cdot \mathcal{N}_\phi \left\{ \cos \left(\beta \left[\phi(x) + (1 - \sigma) \frac{O}{4} \right] + \frac{\Theta}{2} \right) \right\}, \quad (\overline{\Psi}(x) \gamma^5 \Psi(x))_{ren} = \lim_{\varepsilon \rightarrow 0} (-\Lambda^2 \varepsilon^2)^{\overline{\alpha} \overline{\beta} / 4\pi} \overline{\Psi}(x + \varepsilon) \gamma^5 \Psi(x), \quad (140)$$

$$(\overline{\Psi}(x) \gamma^5 \Psi(x))_{ren} \Rightarrow i \frac{\Lambda}{\pi} \left(\frac{\overline{\mu}}{\Lambda} \exp \left\{ -a_0 \frac{\pi}{4} (1 - \sigma)^2 \right\} \right)^{\beta^2 / 4\pi} \mathcal{N}_\phi \left\{ \cos \left(\beta \left[\phi(x) + (1 - \sigma) \frac{O}{4} \right] + \frac{\Theta}{2} \right) \right\}, \quad (141)$$

$$\langle 0 | (\overline{\Psi}(x) \gamma^5 \Psi(x))_{ren} | 0 \rangle \Rightarrow i \frac{\Lambda}{\pi} \left(\frac{\overline{\mu}}{\Lambda} \exp \left\{ -a_0 \frac{\pi}{4} (1 - \sigma)^2 \right\} \right)^{\beta^2 / 4\pi} \cos \frac{\Theta}{2}. \quad (142)$$

Besides $\overline{\mu}$, Λ , and Θ , for $\sigma \neq 1$, these matrix elements (139), (142) depend on additional dimensionless non-physical volume cut-off regularization parameter a_0 , and their final value depend on the order of limits $\overline{\mu} \rightarrow 0$ and $\Lambda \rightarrow \infty$ and on the sign of η (93), that is the sign of g . For the free case: $g = 0 = \eta$, $\beta^2 = 4\pi$, the Λ - dependence disappears as it should be for the free field (55), (56). But for both the HF and free fields the ratio of these condensates (139), (142) remains still equal to $(-i) \tan(\Theta/2)$.

Why the discussed above breaking of the rule (132), remaining meaningless for zero temperature case will be impossible all the more at finite temperature? The reason is the existence of another important formally unitary transformation of the solutions (116), which introduce the two-parametric extension of Oksak solution (87), (88) obeys again the same CAR (9), (10), (94), and the bosonization rule (91) with the same renormalization constant $Z_{(\Psi)}(a)$ (96) for arbitrary σ, ρ , and for $\xi, \xi = \pm$:

$$\mathcal{L}_\rho = \exp \left\{ -\frac{i}{2} \rho Q \bar{Q} Q^{-\xi} \right\} = \exp \left\{ -\frac{i}{8} \rho (O^2 - O_5^2) \right\}, \quad \Psi(x, \sigma, \rho) = \mathcal{L}_\rho^{-1} \Psi(x, \sigma) \mathcal{L}_\rho, \quad (143)$$

$$\Psi_\xi(x, \sigma, \rho) = \mathcal{L}_\rho^{-1} \Psi_\xi(x, \sigma) \mathcal{L}_\rho = K_\sigma^{-1} \Psi_\xi(x, 0, \rho) K_\sigma = \mathcal{N}_\varphi \left\{ e^{R_\xi(x, \sigma, \rho)} \right\} v_\xi(\sigma, \rho), \quad \text{where:} \quad (144)$$

$$R_\xi(x, \sigma, \rho) = -i 2 \sqrt{\pi} \left[\varrho^{-\xi}(x) + \frac{\sigma_0^\xi}{4} \mathcal{W}^{-\xi} + \frac{\sigma_1^\xi}{4} \mathcal{W}^\xi \right] = -i \left[\overline{\alpha} \varphi^{-\xi}(x^{-\xi}) + \overline{\beta} \varphi^\xi(x^\xi) + \frac{\Sigma_0^\xi}{4} Q^{-\xi} + \frac{\Sigma_1^\xi}{4} Q^\xi \right], \quad (145)$$

$$v_\xi(\sigma, \rho) = \left(\frac{\overline{\mu}}{2\pi} \right)^{1/2} \left(\frac{\overline{\mu}}{\Lambda} \right)^{\overline{\beta}^2 / 4\pi} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -\frac{a_0}{32} \left[(\Sigma_0^\xi)^2 + (\Sigma_1^\xi)^2 \right] \right\}, \quad \text{or:} \quad (146)$$

$$v_\xi(\sigma, \rho) = \left(\frac{\overline{\mu}}{2\pi} \right)^{1/2} \left(\frac{\overline{\mu}}{\Lambda} \right)^{\overline{\beta}^2 / 4\pi} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -a_0 \frac{\pi}{8} \left([(\sigma_0^\xi)^2 + (\sigma_1^\xi)^2] \cosh 2\eta - 2\sigma_0^\xi \sigma_1^\xi \sinh 2\eta \right) \right\}, \quad (147)$$

$$\text{with: } \sigma_0^\xi = \sigma_0^\xi(\sigma) = -\xi\sigma, \quad \sigma_1^\xi = \sigma_1^\xi(\rho) = \xi 1 + \rho, \quad \text{and: } \Sigma_0^\xi = \overline{\alpha} \sigma_0^\xi - \overline{\beta} \sigma_1^\xi, \quad \Sigma_1^\xi = \overline{\alpha} \sigma_1^\xi - \overline{\beta} \sigma_0^\xi. \quad (148)$$

Remember that for divergent value of a_0 , as for usual box, the appeared ξ - dependence of last exponential of c - number spinor (147) leads in general to non-physical ξ - dependent and thus non-renormalizable divergences for every components of the field $\Psi_\xi(x, \sigma, \rho)$. Thus appeared ξ - dependence of the last exponential of c - number spinor (147) eliminates for arbitrary ρ only for the case⁸ (124) of Morchio et al. with $\sigma = -\coth 2\eta$, leading to:

$$R_\xi(x, -\coth 2\eta, \rho) \Rightarrow -i \left[\overline{\alpha} \varphi^{-\xi}(x^{-\xi}) + \overline{\beta} \varphi^\xi(x^\xi) + \xi \frac{\pi}{2} \left(\frac{Q^\xi}{\overline{\alpha}} + \frac{Q^{-\xi}}{\overline{\beta}} \right) + \rho \frac{\sqrt{\pi}}{2} \mathcal{W}^\xi \right], \quad (149)$$

$$v_\xi(-\coth 2\eta, \rho) \Rightarrow \left(\frac{\overline{\mu}}{2\pi} \right)^{1/2} \left(\frac{\overline{\mu}}{\Lambda} \right)^{\overline{\beta}^2 / 4\pi} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -a_0 \frac{\pi}{4} \left(\frac{1}{2} + \frac{\overline{\beta}^2}{4\pi} \right) \left[\frac{1}{\sinh^2 2\eta} + \rho^2 \right] \right\}. \quad (150)$$

⁸For $\rho = 0$ in (116) such of dependence was absent for arbitrary σ .

For further reference we write down here also the transformations (116), (144) for the free case:

$$\chi(x, \sigma) = K_\sigma^{-1} \chi(x) K_\sigma, \quad \chi_\xi(x^{-\xi}, \sigma) = K_\sigma^{-1} \chi_\xi(x^{-\xi}) K_\sigma = \mathcal{N}_\varphi \left\{ e^{B_\xi(x^{-\xi}, \sigma)} \right\} u_\xi(\sigma), \quad (151)$$

$$B_\xi(x^{-\xi}, \sigma) = -i \left[2\sqrt{\pi} \varphi^{-\xi}(x^{-\xi}) - \xi \sigma \frac{\sqrt{\pi}}{2} Q^{-\xi} + \xi \frac{\sqrt{\pi}}{2} Q^\xi \right], \quad (152)$$

$$u_\xi(\sigma) = \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -a_0 \frac{\pi}{8} (\sigma^2 + 1) \right\}, \quad (153)$$

$$\chi_\xi(x^{-\xi}, \sigma, \rho) = \mathcal{L}_\rho^{-1} \chi_\xi(x^{-\xi}, \sigma) \mathcal{L}_\rho = K_\sigma^{-1} \chi_\xi(x^{-\xi}, 0, \rho) K_\sigma = \mathcal{N}_\varphi \left\{ e^{B_\xi(x^{-\xi}, \sigma, \rho)} \right\} u_\xi(\sigma, \rho), \quad (154)$$

$$B_\xi(x^{-\xi}, \sigma, \rho) = -i2\sqrt{\pi} \left[\varphi^{-\xi}(x^{-\xi}) + \frac{\sigma_0^\xi}{4} Q^{-\xi} + \frac{\sigma_1^\xi}{4} Q^\xi \right], \quad (155)$$

$$u_\xi(\sigma, \rho) = \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -a_0 \frac{\pi}{8} [(\sigma_0^\xi)^2 + (\sigma_1^\xi)^2] \right\}, \quad (156)$$

which obeys the CAR (8)–(10) and operator bosonization rules (25) with $Z_{(\chi)}(a) = 1$ for arbitrary σ, ρ .

The corresponding VEV of the string of the fields (144)–(148) takes the following form:

$$\begin{aligned} & \left(\Lambda^{\beta^2/4\pi} \sqrt{2\pi} \right)^p \left\langle 0 \left| \prod_{i=1}^p \Psi_{\xi_i}^{(l_i)}(x_i, \sigma, \rho) \right| 0 \right\rangle = \left\langle 0 \left| \mathcal{N}_\varphi \left\{ \exp \left(\sum_{j=1}^p \mathcal{R}_j \right) \right\} \right| 0 \right\rangle \exp \left\{ i\varpi \sum_{i=1}^p l_i - i \frac{\Theta}{4} \sum_{i=1}^p l_i \xi_i \right\} \\ & \cdot \left(\bar{\mu} \exp \left\{ -a_0 \frac{\pi}{4} \left[(1 + \sigma)^2 + (\beta^2/4\pi)^2 \rho^2 \right] \right\} \right)^{\left(\sum_{i=1}^p l_i \right)^2 (\pi/\beta^2)} \\ & \cdot \left(\bar{\mu} \exp \left\{ -a_0 \frac{\pi}{4} \left[(1 - \sigma)^2 + (4\pi/\beta^2)^2 \rho^2 \right] \right\} \right)^{\left(\sum_{i=1}^p l_i \xi_i \right)^2 (\beta^2/16\pi)} \\ & \cdot \exp \left\{ -a_0 \frac{\pi}{4} \rho [\cosh 2\eta + \sigma \sinh 2\eta] \left(\sum_{i=1}^p l_i \xi_i \right) \left(\sum_{j=1}^p l_j \right) \right\} \\ & \cdot \prod_{i < k}^p \left\{ e^{i\pi(\xi_i - \xi_k)} \left[\frac{i(x_i^- - x_k^-) + 0}{i(x_i^+ - x_k^+) + 0} \right]^{\xi_i + \xi_k} \left[i0 \varepsilon(x_i^0 - x_k^0) - (x_i - x_k)^2 \right]^{(4\pi/\beta^2) + \xi_i \xi_k (\beta^2/4\pi)} \right\}^{l_i l_k / 4}. \end{aligned} \quad (157)$$

Here for $\rho \neq 0$ it is impossible to remove the a_0 - dependence from the third line already for any of above mentioned solutions of Mandelstam with $\sigma = \pm 1$, or Morchio et al. with $\sigma = -\coth 2\eta$. It may be only adsorbed into the parameter $\bar{\mu}$ if the first rule (131) fulfills. Now only for the both fulfilled superselection rules (131), (132) this VEV reduces again to above expression in the last line of Eq. (130) or Eqs. (134), (135), which is exactly the last line of Eq. (157), and thus, does not depend on any of the regularization and transformation parameters: $\bar{\mu}$, a_0 , and σ , ρ , ϖ , Θ , and on the any choice of volume cut-off regularization function. Thus, the discarding of superselection rule (132) inevitably spoils the σ , ρ , and Θ - invariance of this n - point fermionic Wightman function and its independence on the parameters $\bar{\mu}$ and a_0 . So, the latters should be fixed by some additional conditions [13], what, however, seems impossible, at least for regularization dependent value of a_0 (see Appendix C).

3 Thirring model for nonzero temperature

3.1 Thermodynamics of ideal 1D gases

From the standard courses [59] it may be easily shown, the equilibrium thermodynamics of the free massless bosons in the 1 -dimension box of length L coincides with that of the free massless spin 1/2 fermions at the same temperature $k_B T = 1/\varsigma$ only for both zero chemical potentials $\mu_{(B)} = \mu_{(F)} = 0$, giving a simplest example of thermal bosonization [30] for pressure P , densities of internal energy \mathcal{U} and entropy S :

$$P_{(B),(F)} = \frac{\mathcal{U}_{(B),(F)}}{L} = \frac{\pi^2}{3\varsigma^2 \hbar c}, \quad \frac{S_{(B),(F)}}{k_B L} = \left(\frac{\partial P_{(B),(F)}}{\partial (1/\varsigma)} \right)_\mu = \frac{2\pi^2}{3\varsigma \hbar c}, \quad \text{however, for given} \quad (158)$$

$$\text{particles densities: } \bar{n}_{(B)} = \frac{N_{(B)}}{L}, \quad \bar{n}_{(F)}^\pm = \frac{N_{(F)}^\pm}{L}, \quad \text{with } h = 2\pi\hbar, \quad c - \text{speed of light:} \quad (159)$$

$$\mu_{(B)}(T, \bar{n}_{(B)}) = \frac{1}{\varsigma} \ln \left(1 - e^{-\bar{n}_{(B)} \varsigma \hbar c / 2} \right), \quad \mu_{(F)}^\pm(T, \bar{n}_{(F)}^\pm) = \pm \frac{1}{\varsigma} \ln \left(e^{\bar{n}_{(F)}^\pm \varsigma \hbar c / 2} - 1 \right). \quad (160)$$

This qualitative “equilibrium” corpuscular picture means, the both systems for the same ς and L have the same P, \mathcal{U}, S , and also another thermodynamic potentials. The condition $\mu_{(B)} = 0$ for arbitrary temperature implies an infinite boson density, $\bar{n}_{(B)} \mapsto \infty$, corresponding to specific case of thermodynamic limit: $N_{(B)} \rightarrow \infty, L \rightarrow \infty$ for the “bosonic picture”. The “equilibrium” fermion pressure (158) of the “fermionic picture” actually is a sum of partial ones of $N_{(F)}^+$ fermions $b(p^1)$ and $N_{(F)}^-$ antifermions $f(p^1)$, defined in (57) and Appendix E, with opposite values of chemical potentials $\mu_{(F)}^\pm = \pm \mu_{(F)}$ [31, 32]:

$$P_{(F)}(T, \mu_{(F)}) = \frac{\mathcal{U}_{(F)}}{L} = P_{(F)}^+(T, \mu_{(F)}^+) + P_{(F)}^-(T, \mu_{(F)}^-) = \frac{\pi^2}{3\varsigma^2 \hbar c} + \frac{\mu_{(F)}^2}{\hbar c}, \quad (161)$$

$$\text{with charge density: } \frac{Q_{(F)}}{L} = \bar{n}_{(F)}^+ - \bar{n}_{(F)}^- = \left(\frac{\partial P_{(F)}}{\partial \mu_{(F)}} \right)_\varsigma = \frac{2\mu_{(F)}}{\hbar c}, \quad \text{where: } Q_{(F)} = \left\langle \left\langle \frac{O_{(\chi)}}{\sqrt{\pi}} \right\rangle \right\rangle_\varsigma, \quad (162)$$

is averaged total charge. For any values of $\mu_{(F)}, \mu_{(B)}$ this recasts the “equilibrium” Gibbs potentials as:

$$\mathcal{G}_{(F)} \equiv \mathcal{U}_{(F)} + P_{(F)}L - TS_{(F)} = N_{(F)}^+ \mu_{(F)}^+ + N_{(F)}^- \mu_{(F)}^- = (N_{(F)}^+ - N_{(F)}^-) \mu_{(F)} = Q_{(F)} \mu_{(F)} = \frac{2L\mu_{(F)}^2}{\hbar c}, \quad (163)$$

$$\mathcal{G}_{(B)} = N_{(B)} \mu_{(B)}. \quad \text{Therefore: } \mathcal{G}_{(F)} \implies \mathcal{G}_{(B)} \longrightarrow 0, \quad (164)$$

only for $\mu_{(F)} = 0$, with $\bar{n}_{(F)}^+ = \bar{n}_{(F)}^- = \bar{n}_{(F)}^0 = 2 \ln 2 / (\varsigma \hbar c)$, i.e. for sector with zero total charge $Q_{(F)} = 0$. Similarly to radiation [59] (see Appendix E for details) the equilibrium pressure of massless particles on the wall originates from their adsorption and emission, thus with $\mu_{(B)} = \mu_{(F)} = 0$. For example, the left wall adsorbs the left moving N_L^+ fermions and N_L^- antifermions, with total charge Q_L , and emits the right moving N_R^+ fermions and N_R^- antifermions, with total charge Q_R [61]. Then, the equilibrium for this wall means $Q_R - Q_L = Q_{5(F)} = 0$. Since for $\mu_{(F)} = 0$ the total charge (162) also vanishes $Q_R + Q_L = Q_{(F)} = 0$, from Eqs. (371), (372), we may conclude, that for such of equilibrium state: $N_{R,L}^+ = N_{R,L}^-$, providing an exact right and left fermions - antifermions pairing into the right and left moving bosons respectively. So, this qualitative “equilibrium” picture admits virtual nonzero fermionic density at finite temperature

$T > 0$, which vanishes only with $T = 0$, corresponding to “clean” fermionic bosonization with zero Fermi energy $\mu_{(F)}^+(0, \bar{n}_{(F)}^+) = \bar{n}_{(F)}^+ hc/2$.

We would like to point out, that for nonzero temperature the previous purely abstract infrared regularization parameter L acquires a physical meaning as a macroscopic thermodynamic parameter (158), (159) [59] of the real or effective “box size” for the thermodynamic system under consideration. So, the corresponding dependence requires additional care, because any function like (324) of the appeared new dimensionless variable L/ς has different limits at $L \rightarrow \infty$ or at $\varsigma \rightarrow \infty$. The volume cut-off regularization function $\Delta(y^1/L)$, (31), (210), divided by $\text{Const} \cdot L$ (with corresponding $\text{Const} = 2$, or $\sqrt{\pi}$, or etc., depends on the functions listed at the Table in Appendix C), acquires a physical meaning of probability density to find the particle at the point y^1 in external field of the “walls” of this “box”. Of course, the physics should not depends on the choice of this regularization function, what is analysed in Appendix C. But it seems, that in any case the **box should have a size L** , – as a physical parameter to provide a possibility of some kind of thermodynamic limit for (31), (43), (50), and (158), (159), as well as for (226), (228), (243), (246), (248), (324) below. From this view point the charge regularization of Refs. [26, 33] (see Appendix C) belongs to the different type since their charge definition has nothing to do with any kind of thermodynamic limit.

3.2 On fermionic tilde conjugation rules

Following to Ojima [48] let us start with simplest fermionic oscillator (for one fixed mode k^1), which has only two normalized states $|0\rangle$ and $|1\rangle$, with energy 0 and ω , annihilated/created by fermionic operators b, b^\dagger : $b|0\rangle = 0$, and $|1\rangle = b^\dagger|0\rangle$, $\{b, b^\dagger\} = 1$, $\{b, b\} = 0$. The thermal vacuum appears as a normalized sum of tensor products of two independent copies of these states: $|0\tilde{0}\rangle = |0\rangle \otimes |\tilde{0}\rangle$, $|1\tilde{1}\rangle = |1\rangle \otimes |\tilde{1}\rangle$, weighted with corresponding Gibbs and relative phase exponential factors [48], so that for $\{b, \tilde{b}^\# \} = 0$, ($\tilde{b}^\# = \tilde{b}, \tilde{b}^\dagger$), omitting for brevity below as above the evident index k^1 , which label all the states and another operators used here, it reads:

$$|0(\varsigma)\rangle_{(F)} = \frac{|0\tilde{0}\rangle + e^{i\Phi} e^{-\varsigma\omega/2} |1\tilde{1}\rangle}{\left[\langle 0\tilde{0}|0\tilde{0}\rangle + e^{-\varsigma\omega} \langle 1\tilde{1}|1\tilde{1}\rangle \right]^{1/2}} \equiv \cos \vartheta(k^1) \left(1 + e^{i\Phi} \tan \vartheta(k^1) b^\dagger \tilde{b}^\dagger \right) |0\tilde{0}\rangle = V_{\vartheta(k^1)(F)}^{-1} |0\tilde{0}\rangle, \quad (165)$$

$$\text{where, for: } \vartheta(k^1) \equiv \vartheta(k^1, \varsigma), \quad \tan^2 \vartheta(k^1, \varsigma) = e^{-\varsigma\omega}, \quad \omega = \omega(k^1): \quad (166)$$

$$V_{\vartheta(k^1)(F)}^{-1} = \exp \left\{ e^{i\Phi} \tan \vartheta(k^1) G_+ \right\} \exp \left\{ -\ln \left(\cos^2 \vartheta(k^1) \right) G_3 \right\} \exp \left\{ -e^{-i\Phi} \tan \vartheta(k^1) G_- \right\}, \quad (167)$$

$$G_+ = b^\dagger \tilde{b}^\dagger, \quad G_- = \tilde{b} b = (G_+)^{\dagger}, \quad G_3 = \frac{1}{2} (b^\dagger b - \tilde{b} \tilde{b}^\dagger) = \frac{1}{2} (b^\dagger b + \tilde{b}^\dagger \tilde{b} - 1), \quad \text{with:} \quad (168)$$

$$[G_+, G_-] = 2G_3, \quad [G_3, G_\pm] = \pm G_\pm, \quad G_\pm = G_1 \pm iG_2, \quad \mathbf{G}^2 = G_3^2 - G_3 + G_+ G_- \Rightarrow j(j+1)\hat{I}, \quad (169)$$

$$\text{thus: } V_{\vartheta(k^1)(F)}^{-1} = \exp \left\{ \vartheta(k^1) \left[e^{i\Phi} G_+ - e^{-i\Phi} G_- \right] \right\} = V_{-\vartheta(k^1)(F)} = V_{\vartheta(k^1)(F)}^{\dagger}, \quad (170)$$

– is a standard form of operator of the coherent state for group $SU(2)$ [62], where the relations (304), (305) are used. This observation allows to identify the algebra (169) as “quasispin” algebra [63], with the “cold” vacuum $|0\tilde{0}\rangle$ as its lowest state for representation with “quasispin” $j = 1/2$, and the state $|1\tilde{1}\rangle$ as the highest one:

$$|0\tilde{0}\rangle \Rightarrow \left| \frac{1}{2}, -\frac{1}{2} \right\rangle, \quad |1\tilde{1}\rangle \Rightarrow \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \quad (171)$$

$$G_3 \left| \frac{1}{2}, \pm \frac{1}{2} \right\rangle = \pm \frac{1}{2} \left| \frac{1}{2}, \pm \frac{1}{2} \right\rangle, \quad G_{\pm} \left| \frac{1}{2}, \pm \frac{1}{2} \right\rangle = 0. \quad (172)$$

The unique arisen arbitrary relative phase Φ reflects now the fact: the quantum state is not the vector, rather the ray. Thus, the thermal vacuum (165), as a coherent state [62], is annihilated by operator:

$$G_-(\varsigma) = V_{\vartheta(k^1)(F)}^{-1} G_- V_{\vartheta(k^1)(F)} = \cos^2 \vartheta G_- + e^{i\Phi} \sin 2\vartheta G_3 - e^{2i\Phi} \sin^2 \vartheta G_+ = \tilde{b}(\varsigma) b(\varsigma), \quad (173)$$

$$\text{as well as by operators: } \begin{cases} b(\varsigma) = V_{\vartheta(k^1)(F)}^{-1} b V_{\vartheta(k^1)(F)} = \cos \vartheta(k^1) b - e^{i\Phi} \sin \vartheta(k^1) \tilde{b}^\dagger, \\ \tilde{b}(\varsigma) = V_{\vartheta(k^1)(F)}^{-1} \tilde{b} V_{\vartheta(k^1)(F)} = \cos \vartheta(k^1) \tilde{b} + e^{i\Phi} \sin \vartheta(k^1) b^\dagger. \end{cases} \quad (174)$$

Up to now $\tilde{b}^\#$ is only notation, which does not define any operation. To fix it as an operation: $\tilde{b}(\varsigma) \mapsto \tilde{\tilde{b}}(\varsigma)$, one should choose the value of Φ . The popular choice $\Phi = 0$ leads to complicated tilde conjugation rules for the fermionic case, different from the bosonic one [35]. The Ojima choice $\Phi = -\pi/2$ gives fermionic rules the same as for bosonic case [48]. We see now, the choice $\Phi = \pi/2$ is also good and, as well as the original Ojima's one, satisfies the properties of antilinear homomorphism and the condition $\tilde{\tilde{b}}(\varsigma) = b(\varsigma)$. It seems convenient for the purposes of bosonization, the tilde operation has the same properties for both Fermi and Bose cases. As a byproduct, we observe a useful interpretation of the thermal vacuum, defined by Bogoliubov transformation (165), as a coherent state, obtained by coherent $SU(2)$ rotation of vacuum states of all Fermi oscillators $|0_{k^1} \tilde{0}_{k^1}\rangle$ for different k^1 as a lowest quasispin states, around one and the same unit vector $\mathbf{u} = (\sin \Phi, \cos \Phi, 0)$, on the different angles $-2\vartheta(k^1, \varsigma)$: $V_{\vartheta(k^1)(F)}^{-1} = \exp [i2\vartheta(k^1, \varsigma) (\mathbf{u} \cdot \mathbf{G})]$ [62].

Analogous picture may be obtained from [48] for thermal Bogoliubov transformation $V_{\vartheta(k^1)(B)}$ of simplest bosonic oscillator for one fixed mode k^1 , leading to connection between the bosonic thermal vacuum and coherent state [62] for the discrete series representation of group $SU(1, 1)$, what is similar to the “small” case discussed in Appendix A (we again omit for brevity the label k^1 for the states and another operators):

$$|0(\varsigma)\rangle_{(B)} = \frac{\sum_{n=0}^{\infty} e^{i\Phi_n} e^{-n\varsigma\omega/2} |n\tilde{n}\rangle}{\left[\sum_{n=0}^{\infty} e^{-n\varsigma\omega} \langle n\tilde{n}|n\tilde{n}\rangle \right]^{1/2}} \xrightarrow{\Phi_n \mapsto n\Phi} \frac{1}{\cosh \vartheta(k^1)} \exp \left(e^{i\Phi} \tanh \vartheta(k^1) a^\dagger \tilde{a}^\dagger \right) |0\tilde{0}\rangle = V_{\vartheta(k^1)(B)}^{-1} |0\tilde{0}\rangle, \quad (175)$$

$$[a, a^\dagger] = [\tilde{a}, \tilde{a}^\dagger] = 1, \quad [a, \tilde{a}^\#] = 0, \quad |n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle, \quad |\tilde{n}\rangle = \frac{(\tilde{a}^\dagger)^n}{\sqrt{n!}} |\tilde{0}\rangle, \quad \tanh^2 \vartheta(k^1, \varsigma) = e^{-\varsigma\omega(k^1)}, \quad (176)$$

$$Y_+ = a^\dagger \tilde{a}^\dagger, \quad Y_- = \tilde{a} a = (Y_+)^{\dagger}, \quad Y_0 = \frac{1}{2} (a^\dagger a + \tilde{a} \tilde{a}^\dagger) = \frac{1}{2} (1 + a^\dagger a + \tilde{a}^\dagger \tilde{a}), \quad (177)$$

$$[Y_-, Y_+] = 2Y_0, \quad [Y_0, Y_{\pm}] = \pm Y_{\pm}, \quad C_2 = Y_0^2 - Y_0 - Y_+ Y_- = \frac{1}{4} \left[-1 + (a^\dagger a - \tilde{a}^\dagger \tilde{a})^2 \right] \Rightarrow \kappa(\kappa - 1) \hat{I}, \quad (178)$$

$$C_2 |\kappa, \nu\rangle = \kappa |\kappa, \nu\rangle, \quad Y_0 |\kappa, \nu\rangle = \nu |\kappa, \nu\rangle, \quad \nu = \kappa + m, \quad \nu \Rightarrow \kappa = \frac{1}{2}, \quad |0\tilde{0}\rangle \Rightarrow \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \quad (179)$$

$$V_{\vartheta(k^1)(B)}^{-1} = \exp \left\{ e^{i\Phi} \tanh \vartheta(k^1) Y_+ \right\} \exp \left\{ -\ln(\cosh^2 \vartheta(k^1)) Y_0 \right\} \exp \left\{ -e^{-i\Phi} \tanh \vartheta(k^1) Y_- \right\}, \quad (180)$$

$$\text{thus: } V_{\vartheta(k^1)(B)}^{-1} = \exp \left\{ \vartheta(k^1) \left[e^{i\Phi} Y_+ - e^{-i\Phi} Y_- \right] \right\} = V_{-\vartheta(k^1)(B)} = V_{\vartheta(k^1)(B)}^{\dagger}, \quad \text{and so on.} \quad (181)$$

However for this case the numerator in (175) contains a countable number of terms with countable number of arbitrary phases Φ_n [48]. By means of the relations (304), (305), the coherent state (180), (181) would be obtained now only for countable number of coherent choices, $\Phi_n \mapsto n\Phi$, $n = 0, 1, 2, \dots$, already for every one simplest bosonic oscillator for given k^1 only. We did not find a reason to prefer this choice instead of the usual one $\Phi_n = 0$ [48], again fixes the tilde-operation as antilinear homomorphism with the condition $\tilde{\tilde{a}}(\varsigma) = a(\varsigma)$ [35]. Note the inequivalent vacuum again appears here as a coherent state, as well as in (53) [47] or as for the case of the c - number field's shift in Refs. [35, 47].

Analogously to (306)–(312), the total thermal transformation is given by infinite product of operators (181) for $a, \tilde{a} = a(k^1), \tilde{a}(k^1)$, and the transformed vacuum state is an infinite product of one mode states similar to (312):

$$\text{with: } a(k^1) \Rightarrow \frac{c(k^1)}{\sqrt{2k^0 L}}, \quad \text{as: } \mathcal{V}_{\vartheta(B)}^{-1} = \prod_{k^1=-\infty}^{\infty} \mathcal{V}_{\vartheta(k^1)(B)}^{-1} = \exp\{-\mathcal{X}_{\vartheta}\} = \mathcal{V}_{\vartheta(B)}^{\dagger}, \quad \mathcal{V}_{\vartheta(B)} = e^{\mathcal{X}_{\vartheta}}, \quad (182)$$

$$\mathcal{X}_{\vartheta} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dq^1}{2q^0} \vartheta(q^1, \varsigma) \left(\tilde{c}(q^1) c(q^1) - c^{\dagger}(q^1) \tilde{c}^{\dagger}(q^1) \right) = \tilde{\mathcal{X}}_{\vartheta} = -\mathcal{X}_{\vartheta}^{\dagger}, \quad |0(\varsigma)\rangle = \mathcal{V}_{\vartheta(B)}^{-1} |0\tilde{0}\rangle, \quad (183)$$

$$\begin{aligned} \mathcal{V}_{\vartheta(B)}^{-1} = & \exp \left\{ \frac{1}{2\pi} \int_{-\infty}^{+\infty} dk^1 \tanh \vartheta(k^1, \varsigma) \frac{c^{\dagger}(k^1) \tilde{c}^{\dagger}(k^1)}{2k^0} \right\} \\ & \cdot \exp \left\{ -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{dk^1}{2k^0} \ln \left(\cosh^2 \vartheta(k^1, \varsigma) \right) \frac{1}{2} \left(c^{\dagger}(k^1) c(k^1) + \tilde{c}(k^1) \tilde{c}^{\dagger}(k^1) \right) \right\} \\ & \cdot \exp \left\{ -\frac{1}{2\pi} \int_{-\infty}^{+\infty} dk^1 \tanh \vartheta(k^1, \varsigma) \frac{\tilde{c}(k^1) c(k^1)}{2k^0} \right\}, \end{aligned} \quad (184)$$

$$\begin{aligned} c(k^1; [\pm]\varsigma) &= \mathcal{V}_{\vartheta(B)}^{\mp 1} c(k^1) \mathcal{V}_{\vartheta(B)}^{\pm 1} = \cosh \vartheta c(k^1) [\mp] \sinh \vartheta \tilde{c}^{\dagger}(k^1), & c(k^1; [+]\varsigma) |0(\varsigma)\rangle &= 0, \\ \tilde{c}(k^1; [\pm]\varsigma) &= \mathcal{V}_{\vartheta(B)}^{\mp 1} \tilde{c}(k^1) \mathcal{V}_{\vartheta(B)}^{\pm 1} = \cosh \vartheta \tilde{c}(k^1) [\mp] \sinh \vartheta c^{\dagger}(k^1), & \tilde{c}(k^1; [+]\varsigma) |0(\varsigma)\rangle &= 0, \end{aligned} \quad (185)$$

$$\begin{aligned} \left[c(k^1; [\pm]\varsigma), c^{\dagger}(q^1; [\pm]\varsigma) \right] &= (2\pi) (2k^0) \delta(k^1 - q^1), & \tanh^2 \vartheta(k^1, \varsigma) &= e^{-\varsigma k^0}, \quad k^0 = |k^1|. \\ \left[\tilde{c}(k^1; [\pm]\varsigma), \tilde{c}^{\dagger}(q^1; [\pm]\varsigma) \right] &= (2\pi) (2k^0) \delta(k^1 - q^1), \end{aligned} \quad (186)$$

Thus, for “hot” pseudoscalar field with respect to the “hot” vacuum $|0(\varsigma)\rangle$ (183), (185) one finds [35, 48]:

$$\phi(x; [+]\varsigma) = \mathcal{V}_{\vartheta(B)}^{-1} \phi(x) \mathcal{V}_{\vartheta(B)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \left[c(k^1; [+]\varsigma) e^{-i(kx)} + c^{\dagger}(k^1; [+]\varsigma) e^{+i(kx)} \right]. \quad (187)$$

The meaning of the label $[\pm]$ of the field is explained below.

3.3 Temperature equation of motion. “Hot” and “cold” thermofields

So in the framework of thermofield dynamics [35], at finite temperature it is necessary to double the number of degrees of freedom by providing all the fields Ψ with their tilde partners $\tilde{\Psi}$. According to

[35], the resulting theory will be determined by the Hamiltonian $\hat{H}[\Psi, \tilde{\Psi}] = H[\Psi] - \tilde{H}[\tilde{\Psi}]$, where $\tilde{H}[\tilde{\Psi}] = H^*[\tilde{\Psi}^*]$, with $H[\Psi] = H_{0[\Psi]}(x^0) + H_{I[\Psi]}(x^0)$ given by (5)-(7), whence for Thirring model: $\tilde{H}_{I[\tilde{\Psi}]} = H_{I[\tilde{\Psi}]}$, and $\tilde{H}_{0[\tilde{\Psi}]} = -H_{0[\tilde{\Psi}]}$. Though the substitution like (3), (174), for the free massless Dirac thermofields, $\chi(x) \mapsto \chi(x, \varsigma)$, also does not change the form [35] of the free operator: $\hat{H}_0[\chi, \tilde{\chi}] = H_0[\chi] - \tilde{H}_0[\tilde{\chi}]$, these free fields, as explained above, generally speaking, are not now the physical fields of this QFT model [32, 33], and each of the terms in $\hat{H}[\Psi, \tilde{\Psi}]$ must be equivalent in a weak sense to the free Hamiltonian of massless (pseudo) scalar fields $(\phi(x))$, $\varphi(x)$, at least at zero temperature, $T = 0$ [33, 35].

For any functional $\mathcal{F}[\Psi]$ of HF Ψ in the representation of given physical fields $\psi(x)$, i.e. for dynamical mapping $\Psi(x) = \Upsilon[\psi(x)]$ at zero temperature, being interested in the matrix elements for the thermal vacuum of the type:

$$\langle 0(\varsigma) | \mathcal{F}[\Psi(x)] | 0(\varsigma) \rangle = \langle 0\tilde{0} | \mathcal{V}_\vartheta \mathcal{F}[\Psi(x)] \mathcal{V}_\vartheta^{-1} | 0\tilde{0} \rangle = \langle 0\tilde{0} | \mathcal{F}[\mathcal{V}_\vartheta \Psi(x) \mathcal{V}_\vartheta^{-1}] | 0\tilde{0} \rangle, \quad (188)$$

$$\text{we come to formal mapping: } \mathcal{V}_\vartheta \Psi(x) \mathcal{V}_\vartheta^{-1} = \Psi(x, [-]\varsigma) = \Upsilon[\mathcal{V}_\vartheta \psi(x) \mathcal{V}_\vartheta^{-1}] = \Upsilon[\psi(x, [-]\varsigma)], \quad (189)$$

$$\text{onto the “cold” physical thermofield: } \psi(x, [-]\varsigma) = \mathcal{V}_\vartheta \psi(x) \mathcal{V}_\vartheta^{-1}, \quad (190)$$

essentially with the same coefficient functions, as for the initial DM $\Psi(x) = \Upsilon[\psi(x)]$, that, contrary to [35, 48], thus transferring so all the temperature dependence from the “hot” vacuum state (1), (165), (175), (183), (185), onto these “cold” physical thermofields. However, to compute the matrix element (188) it is necessary to substitute into the r.h.s. of (188), (189) the “cold” physical thermofields (190) again in terms of the initial physical fields $\psi(x)$ via obtained from (190) their linear combinations (185)[-], analogous (but not the same!) to Eqs. (3), (174), (185)[+], and reorder again the so obtained operator with respect to the initial physical fields $\psi(x)$. The same operations also convert the formal mapping (189) into temperature dependent DM with respect to the “cold” vacuum $|0\tilde{0}\rangle$, and precisely in such of sense we call further the r.h.s. of (189) again as a new DM $\hat{\Upsilon}$, or e.g. $c(k^1) \mapsto b(k^1), f(k^1)$:

$$\Psi(x, [-]\varsigma) = \Upsilon[\psi(x, [-]\varsigma)] = \Upsilon[\mathcal{V}_\vartheta \psi(x) \mathcal{V}_\vartheta^{-1}] \implies \hat{\Upsilon}[[-\varsigma; \psi(x)] = \hat{\Upsilon}[[-\varsigma; c(k^1), \tilde{c}(k^1)]]. \quad (191)$$

On the contrary, the standard computation way [35, 48] implies the substitution of the inverse to (3), (174), (185)[+] linear expressions of physical fields $\psi(x) = \mathcal{V}_\vartheta \psi(x, [+]\varsigma) \mathcal{V}_\vartheta^{-1}$ in terms of the “hot” physical thermofields $\psi(x, [+]\varsigma) = \mathcal{V}_\vartheta^{-1} \psi(x) \mathcal{V}_\vartheta$, given by the (3), (174), (185)[+], into the l.h.s. of (188) and reordering the so obtained operator with respect to this “hot” physical thermofield i.e. with respect to the thermal “hot” – vacuum (1), (165), (175), (183), (185). Of course, the same operations give the new DM $\hat{\Upsilon}$ for the initial HF with respect to this “hot” thermal vacuum (185) [35]:

$$\Psi(x) = \Upsilon[\psi(x)] = \Upsilon[\mathcal{V}_\vartheta \psi(x, [+]\varsigma) \mathcal{V}_\vartheta^{-1}] \implies \hat{\Upsilon}[[+\varsigma; \psi(x, [+]\varsigma)] = \hat{\Upsilon}[[+\varsigma; c(k^1, [+]\varsigma), \tilde{c}(k^1, [+]\varsigma)]]. \quad (192)$$

We would like to point out, that this field does not equal to $\Psi(x, [+]\varsigma) = \mathcal{V}_\vartheta^{-1} \Psi(x) \mathcal{V}_\vartheta$, which will appear below as a byproduct of our further consideration⁹ similar to (191). So, to avoid some ambiguities [51,

⁹reading as: $\Psi(x, [+]\varsigma) = \Upsilon[\psi(x, [+]\varsigma)] = \Upsilon[\mathcal{V}_\vartheta^{-1} \psi(x) \mathcal{V}_\vartheta] \implies \hat{\Upsilon}[[+\varsigma; \psi(x)] = \hat{\Upsilon}[[+\varsigma; c(k^1), \tilde{c}(k^1)]]$, i.e. again with respect to the “cold” vacuum $|0\tilde{0}\rangle$.

52, 53] one should carefully distinguish the use of “hot” and “cold” physical thermofields $\psi(x, [\pm]\varsigma)$ with respect to corresponding vacua.

The kinematic independence of fermionic tilde-conjugate fields $\tilde{\Psi}$ for $T = 0$ means:

$$\left\{ \Psi_\xi(x), \tilde{\Psi}_{\xi'}^\#(y) \right\} \Big|_{x^0=y^0} = 0, \quad \left\{ \Psi_\xi(x), \tilde{\Psi}_{\xi'}^\#(y) \right\} \Big|_{(x-y)^2 < 0} = 0, \quad (193)$$

and corresponds to above independence of their Hamiltonians and their HEqs for $T = 0$. This allows to consider a solution only for the one of them. Since the thermal transformations $\mathcal{V}_{\vartheta(F)}$, $\mathcal{V}_{\vartheta(B)}$ are not depend on coordinates and time, they can be applied directly to zero temperature HEq of Thirring model (13), (14), resulting to the same HEqs for the new HF (189), with the same kinematic independence condition (193) for finite temperature, where we omit for brevity the label $[\pm]$ of $\Psi(x; [\pm]\varsigma)$, where it is not important:

$$i\partial_0 \Psi(x, \varsigma) = \left[\Psi(x, \varsigma), \hat{H}[\Psi, \tilde{\Psi}] \right] = \left[E(P^1) + g\gamma^0 \gamma_\nu J_\nu^\Psi(x, \varsigma) \right] \Psi(x, \varsigma), \quad (194)$$

$$\text{or: } 2\partial_\xi \Psi_\xi(x, \varsigma) = -igJ_{(\Psi)}^{-\xi}(x, \varsigma) \Psi_\xi(x, \varsigma), \quad \xi = \pm, \quad (195)$$

$$\text{so: } 2\partial_\xi \tilde{\Psi}_\xi(x, \varsigma) = +ig\tilde{J}_{(\tilde{\Psi})}^{-\xi}(x, \varsigma) \tilde{\Psi}_\xi(x, \varsigma), \quad \xi = \pm, \quad (196)$$

$$\left\{ \Psi_\xi(x, \varsigma), \tilde{\Psi}_{\xi'}^\#(y, \varsigma) \right\} \Big|_{x^0=y^0} = 0, \quad \left\{ \Psi_\xi(x, \varsigma), \tilde{\Psi}_{\xi'}^\#(y, \varsigma) \right\} \Big|_{(x-y)^2 < 0} = 0, \quad (197)$$

– for each ξ -component of the fields $\Psi_\xi(x, \varsigma)$, $\tilde{\Psi}_\xi(x, \varsigma)$, that are also formally related to the corresponding current components as:

$$J_{(\Psi)}^\xi(x, \varsigma) = J_{(\Psi)}^0(x, \varsigma) + \xi J_{(\Psi)}^1(x, \varsigma) \longmapsto 2\Psi_\xi^\dagger(x, \varsigma) \Psi_\xi(x, \varsigma), \quad \xi = \pm, \quad (198)$$

and the same for tilde-conjugate currents and fields. Thus, to integrate these HEqs we can sequentially repeat all the previous zero temperature steps of the previous section, [54, 55, 56]. Applying the same arguments based on the currents conservation: $\partial_\xi J_{(\Psi)}^\xi(x, \varsigma) = 0$, $\xi = \pm$, we come to the same linearization, renormalization and bosonization conditions in the sense of weak equality. We reproduce briefly all these steps in the next subsections to outline the main differences. The first one is due to appearance of tilde-conjugate fields. By virtue of (174), (185), the tilde-conjugation rule for corresponding DM (191), (192) takes the most simple form in momentum representation [35], in terms of initial annihilation/creation operators $c(k^1)$, $b(k^1)$, $f(k^1)$, etc. for zero temperature.

3.4 Linearization of the Heisenberg equation

From the equation (194) and anticommutation relations for the field operators (8)–(10) it follows again that in the canonical equation of motion for the “total current” operator (12), (198) from the right hand side of HEq. (194):

$$i\partial_0 \gamma^0 \gamma_\nu J_\nu^\Psi(x, \varsigma) - \left[\gamma^0 \gamma_\nu J_\nu^\Psi(x, \varsigma), H_{0[\Psi]}(x^0, \varsigma) \right] = iI \partial_\mu J_\mu^\Psi(x, \varsigma) + i\gamma^5 \epsilon_{\mu\nu} \partial^\mu J_\nu^\Psi(x, \varsigma) = 0, \quad (199)$$

$$i\partial_0 \gamma^0 \gamma_\nu J_\nu^\Psi(x, \varsigma) - \left[\gamma^0 \gamma_\nu J_\nu^\Psi(x, \varsigma), H_{0[\Psi]}(x^0, \varsigma) \right] = \left[\gamma^0 \gamma_\nu J_\nu^\Psi(x, \varsigma), H_{I[\Psi]}(x^0, \varsigma) \right] = 0, \quad (200)$$

due to currents conservation $\partial_\xi J_{(\Psi)}^\xi(x, \varsigma) = 0$, again vanishes the contribution of the commutator with the interaction Hamiltonian $H_{I(\Psi)}(x^0, \varsigma)$. Therefore, the temporal evolution of this “total current” will be

again described by a free Hamiltonian $H_{0(\chi)}(x^0, \varsigma)$, quadratic on some free trial physical Dirac fields $\chi(x, \varsigma)$, furnished by the same anti-commutation relations (8)–(10), (193), (197), and by the same conservation laws for corresponding currents $J_{(\chi)}^\nu(x, \varsigma)$, $J_{(\chi)}^{5\nu}(x, \varsigma)$, given by Eqs. (12), (198) with $\Psi(x, \varsigma) \mapsto \chi(x, \varsigma)$:

$$i\partial_0\gamma^0\gamma_\nu J_{(\chi)}^\nu(x, \varsigma) - \left[\gamma^0\gamma_\nu J_{(\chi)}^\nu(x, \varsigma), H_{0[\chi]}(x^0, \varsigma) \right] = iI \partial_\mu J_{(\chi)}^\mu(x, \varsigma) + i\gamma^5 \epsilon_{\mu\nu} \partial^\mu J_{(\chi)}^\nu(x, \varsigma) = 0. \quad (201)$$

So, as in above section, the allowance of a possible contribution into (200) of the Schwinger terms would be premature, leading to contradiction with the vector and pseudovector currents conservation conditions. As above, the Heisenberg current operators appearing in (194), (195) acquire precise operator meaning – with non-vanishing Schwinger term – again only after the choice of the representation space [4], [40], [49] for anticommutation relations (8)–(10), (193), (197), and subsequent reduction in this representation to the normal ordered form by means of renormalization, for example, again via point-splitting and subtraction of the VEV [33], but taken now with respect to the initial “cold” vacuum $|0\tilde{0}\rangle$:

$$J_{(\Psi)}^0(x, \varsigma) \mapsto \lim_{\tilde{\varepsilon} \rightarrow 0} \hat{J}_{(\Psi)}^0(x; \tilde{\varepsilon}, \varsigma) = \hat{J}_{(\Psi)}^0(x, \varsigma), \quad J_{(\Psi)}^1(x, \varsigma) \mapsto \lim_{\varepsilon \rightarrow 0} \hat{J}_{(\Psi)}^1(x; \varepsilon, \varsigma) = \hat{J}_{(\Psi)}^1(x, \varsigma), \quad (202)$$

$$\text{where at first: } \tilde{\varepsilon}^0 = \varepsilon^1 \rightarrow 0, \quad \text{when: } \tilde{\varepsilon}^1 = \varepsilon^0, \quad \varepsilon^2 = -\tilde{\varepsilon}^2 > 0, \quad (203)$$

$$\text{for: } \hat{J}_{(\Psi)}^\nu(x; a, \varsigma) = Z_{(\Psi)}^{-1}(a) \left[\overline{\Psi}(x+a, \varsigma) \gamma^\nu \Psi(x, \varsigma) - \langle 0\tilde{0} | \overline{\Psi}(x+a, \varsigma) \gamma^\nu \Psi(x, \varsigma) | 0\tilde{0} \rangle \right], \quad (204)$$

and accordingly for every ξ - component (198). The renormalization “constant” $Z_{(\Psi)}(a)$ is defined below in Eq. (297). With these remarks, observations (200), (201) again allow to identify at least in a weak sense, the Heisenberg operator of “total current” on the r.h.s. of Eq. (194), defined by Eqs. (12), (200), with that operator, defined by Eqs. (12), (201) for the free massless trial physical Dirac fields $\chi(x, \varsigma)$ and renormalized in the sense of normal form (202)–(204) up to an unknown yet constant β :

$$\gamma^0\gamma_\nu J_{(\Psi)}^\nu(x, \varsigma) \xrightarrow{w} \frac{\beta}{2\sqrt{\pi}} \gamma^0\gamma_\nu \hat{J}_{(\chi)}^\nu(x, \varsigma), \quad \text{where again:} \quad (205)$$

$$\hat{J}_{(\chi)}^\nu(x, \varsigma) = \lim_{\varepsilon, (\tilde{\varepsilon}) \rightarrow 0} \hat{J}_{(\chi)}^\nu(x; \varepsilon(\tilde{\varepsilon}), \varsigma) \equiv J_{(\chi)}^\nu(x, \varsigma), \quad \text{for: } Z_{(\chi)}(a) = 1, \quad (206)$$

what leads again to the linearization of both equations (194), (195) in the representation of these free trial physical massless Dirac fields $\chi(x, \varsigma)$.

3.5 Thermal bosonization and scalar fields

Again, the use of BP also simplifies integration of linearized HEqs (194). Being again a formal consequence of the current conservation conditions (199) only, the bosonization rules have, generally speaking, the sense of weak equalities only for the current operator in the normal-ordered form (202)–(204), that already implies a choice of certain representations of (anti-) commutation relations (8)–(10), (193), (197), and (213) below. However, for the free massless fields $\chi(x, \varsigma)$, $\varphi(x, \varsigma)$, $\phi(x, \varsigma)$, this choice is again carried out “almost automatically”. This, due to the linearization condition (205), (206), again becomes enough for our purposes, since for the free fields these relationships appear as operator equalities [33]:

$$\hat{J}_{(\chi)}^\mu(x, \varsigma) = \frac{1}{\sqrt{\pi}} \partial^\mu \varphi(x, \varsigma) = -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \phi(x, \varsigma), \quad \hat{J}_{(\chi)}^{-\xi}(x, \varsigma) = \frac{2}{\sqrt{\pi}} \partial_\xi \varphi^\xi(x^\xi, \varsigma), \quad (207)$$

where the thermofields $\varphi(x, \varsigma)$ and $\phi(x, \varsigma)$ are defined in (221) below as unitarily inequivalent representations of the massless scalar and pseudoscalar Klein-Gordon fields: $\partial_\mu \partial^\mu \varphi(x, \varsigma) = 0$, and $\partial_\mu \partial^\mu \phi(x, \varsigma) = 0$, and are taken again mutually dual and coupled by the symmetric integral relations:

$$\left. \begin{array}{l} \phi(x, \varsigma) \\ \varphi(x, \varsigma) \end{array} \right\} = -\frac{1}{2} \int_{-\infty}^{\infty} dy^1 \varepsilon(x^1 - y^1) \partial_0 \left\{ \begin{array}{l} \varphi(y^1, x^0, \varsigma) \\ \phi(y^1, x^0, \varsigma) \end{array} \right\}, \quad (208)$$

that again implies the solitonic type of asymptotical conditions:

$$\varphi(-\infty, x^0, \varsigma) + \varphi(\infty, x^0, \varsigma) = 0, \quad \phi(-\infty, x^0, \varsigma) + \phi(\infty, x^0, \varsigma) = 0, \quad (209)$$

with the conserved charges corresponding to these fields, as:

$$\left. \begin{array}{l} O(\varsigma) \\ O_5(\varsigma) \end{array} \right\} = \lim_{L \rightarrow \infty} \int_{-\infty}^{\infty} dy^1 \Delta \left(\frac{y^1}{L} \right) \partial_0 \left\{ \begin{array}{l} \varphi(y^1, x^0, \varsigma) \\ \phi(y^1, x^0, \varsigma) \end{array} \right\} \xrightarrow{\Delta=1} \left\{ \begin{array}{l} \phi(-\infty, x^0, \varsigma) - \phi(\infty, x^0, \varsigma) \\ \varphi(-\infty, x^0, \varsigma) - \varphi(\infty, x^0, \varsigma) \end{array} \right\}. \quad (210)$$

The right and left thermofields $\varphi^\xi(x^\xi, \varsigma)$ and their charges $Q^\xi(\varsigma)$ are defined again by the same linear combinations (32)–(33), [33]:

$$\varphi^\xi(x^\xi, \varsigma) = \frac{1}{2} [\varphi(x, \varsigma) - \xi \phi(x, \varsigma)], \quad \text{for: } \xi = \pm, \quad (211)$$

$$Q^\xi(\varsigma) = \frac{1}{2} [O(\varsigma) - \xi O_5(\varsigma)] = \xi \varphi^\xi(x^0 + \xi \infty, \varsigma) - \xi \varphi^\xi(x^0 - \xi \infty, \varsigma) = \pm 2 \varphi^\xi(x^0 \pm \infty, \varsigma), \quad (212)$$

The fields $\varphi(x, \varsigma)$, $\phi(x, \varsigma)$, $\varphi^\xi(x^\xi, \varsigma)$ and their charges obey the same commutation relations (34)–(38), that are not depended on temperature, for example:

$$[\varphi(x, \varsigma), \partial_0 \varphi(y, \varsigma)]|_{x^0=y^0} = [\phi(x, \varsigma), \partial_0 \phi(y, \varsigma)]|_{x^0=y^0} = i \delta(x^1 - y^1), \quad (213)$$

$$[\varphi(x, \varsigma), \varphi(y, \varsigma)] = [\phi(x, \varsigma), \phi(y, \varsigma)] = -i \frac{\varepsilon(x^0 - y^0)}{2} \theta((x - y)^2), \quad (214)$$

$$[\varphi^\xi(s, \varsigma), \varphi^{\xi'}(\tau, \varsigma)] = -\frac{i}{4} \varepsilon(s - \tau) \delta_{\xi, \xi'}, \quad [\varphi^\xi(s, \varsigma), Q^{\xi'}(\varsigma)] = \frac{i}{2} \delta_{\xi, \xi'}. \quad (215)$$

Moreover, the similar commutation relations (but not the same!) take place for their tilde-partners:

$$[\tilde{\varphi}(x, \varsigma), \partial_0 \tilde{\varphi}(y, \varsigma)]|_{x^0=y^0} = [\tilde{\phi}(x, \varsigma), \partial_0 \tilde{\phi}(y, \varsigma)]|_{x^0=y^0} = -i \delta(x^1 - y^1), \quad (216)$$

$$[\tilde{\varphi}(x, \varsigma), \tilde{\varphi}(y, \varsigma)] = [\tilde{\phi}(x, \varsigma), \tilde{\phi}(y, \varsigma)] = +i \frac{\varepsilon(x^0 - y^0)}{2} \theta((x - y)^2), \quad (217)$$

$$[\tilde{\varphi}^\xi(s, \varsigma), \tilde{\varphi}^{\xi'}(\tau, \varsigma)] = +\frac{i}{4} \varepsilon(s - \tau) \delta_{\xi, \xi'}, \quad [\tilde{\varphi}^\xi(s, \varsigma), \tilde{Q}^{\xi'}(\varsigma)] = -\frac{i}{2} \delta_{\xi, \xi'}, \quad (218)$$

that remain kinematically independent in the sense of Eqs. (193), (197), also at finite temperature:

$$[\varphi^\xi(s, \varsigma), \tilde{\varphi}^{\xi'}(\tau, \varsigma)] = 0, \quad [\varphi^\xi(s, \varsigma), \tilde{Q}^{\xi'}(\varsigma)] = 0, \quad [Q^\xi(\varsigma), \tilde{Q}^{\xi'}(\varsigma)] = 0. \quad (219)$$

So, up to now we cannot distinguish the “hot” and “cold” physical thermofields.

The kinematic independence of the tilde-partners fails and the difference between the “hot” and “cold” physical thermofields appears on going to the “frequency” parts of corresponding physical fields $\varphi^{\xi(\pm)}(x^\xi, \varsigma)$, and their charges $Q^{\xi(\pm)}(\varsigma)$ with respect to any of chosen vacuum state. In particular, it manifests itself in the commutators of annihilation (+) and creation (−) (frequency) parts, defined according to (44), by annihilation and creation operators for the one and the same initial “cold” vacuum $|0\tilde{0}\rangle$: $c(k^1)|0\tilde{0}\rangle = \tilde{c}(k^1)|0\tilde{0}\rangle = 0$, for both the “hot” [+], and “cold” [−] physical thermofields by making use of Eqs. (183)–(187), in the form:

$$|0(\varsigma)\rangle = \mathcal{V}_{\vartheta(B)}^{-1}|0\tilde{0}\rangle \equiv \mathcal{V}_{(B)}[-\vartheta]|0\tilde{0}\rangle, \quad \vartheta = \vartheta(k^1; \varsigma), \quad \tanh^2 \vartheta(k^1; \varsigma) = e^{-\varsigma k^0}, \quad (220)$$

$$\varphi(x; [\pm]\varsigma) = \mathcal{V}_{\vartheta(B)}^{\mp 1} \varphi(x) \mathcal{V}_{\vartheta(B)}^{\pm 1} \implies \varphi^{(+)}(x; [\pm]\varsigma) + \varphi^{(-)}(x; [\pm]\varsigma), \quad (221)$$

and so on for all other free physical (pseudo) scalar fields and charges $\phi(x), \omega(x), \Omega(x), O, O_5, \mathcal{W}^\xi, \dots$, with corresponding Fourier expansions and commutators. Below we put corresponding \pm into respective brackets, and $k^0 = |k^1|$:

$$\varphi^{\xi(+)}(x^\xi; [\pm]\varsigma) = -\frac{\xi}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \theta(-\xi k^1) \left[\cosh \vartheta c(k^1) e^{-ik^0 x^\xi} \mp \sinh \vartheta \tilde{c}(k^1) e^{ik^0 x^\xi} \right], \quad (222)$$

$$\varphi^{\xi(-)}(x^\xi; [\pm]\varsigma) = \left\{ \varphi^{\xi(+)}(x^\xi; [\pm]\varsigma) \right\}^\dagger, \quad (223)$$

$$\tilde{\varphi}^{\xi(+)}(x^\xi; [\pm]\varsigma) = -\frac{\xi}{2\pi} \int_{-\infty}^{\infty} \frac{dk^1}{2k^0} \theta(-\xi k^1) \left[\cosh \vartheta \tilde{c}(k^1) e^{ik^0 x^\xi} \mp \sinh \vartheta c(k^1) e^{-ik^0 x^\xi} \right], \quad (224)$$

$$\tilde{\varphi}^{\xi(-)}(x^\xi; [\pm]\varsigma) = \left\{ \tilde{\varphi}^{\xi(+)}(x^\xi; [\pm]\varsigma) \right\}^\dagger, \quad (225)$$

$$Q^{\xi(+)}([\pm]\varsigma) = \lim_{L \rightarrow \infty} i \frac{\xi}{2} \int_{-\infty}^{\infty} dk^1 \theta(-\xi k^1) \left[\cosh \vartheta c(k^1) e^{-ik^0 \hat{x}^0} \pm \sinh \vartheta \tilde{c}(k^1) e^{ik^0 \hat{x}^0} \right] \delta_L(k^1), \quad (226)$$

$$Q^{\xi(-)}([\pm]\varsigma) = \left\{ Q^{\xi(+)}([\pm]\varsigma) \right\}^\dagger, \quad (227)$$

$$\tilde{Q}^{\xi(+)}([\pm]\varsigma) = \lim_{L \rightarrow \infty} -i \frac{\xi}{2} \int_{-\infty}^{\infty} dk^1 \theta(-\xi k^1) \left[\cosh \vartheta \tilde{c}(k^1) e^{ik^0 \hat{x}^0} \pm \sinh \vartheta c(k^1) e^{-ik^0 \hat{x}^0} \right] \delta_L(k^1), \quad (228)$$

$$\tilde{Q}^{\xi(-)}([\pm]\varsigma) = \left\{ \tilde{Q}^{\xi(+)}([\pm]\varsigma) \right\}^\dagger. \quad (229)$$

Here, as before in (50), the \hat{x}^0 – dependence of charge frequency parts (226), (228) is fictitious and non-physical. It is an artifact of space regularization (210) and should be eliminated at the end of calculation.

Only for “hot” [+] thermofields one has:

$$\langle 0(\varsigma) | \varphi^\xi(s; [+]\varsigma) \varphi^{\xi'}(\tau; [+]\varsigma) | 0(\varsigma) \rangle = \langle 0 | \varphi^\xi(s) \varphi^{\xi'}(\tau) | 0 \rangle = \quad (230)$$

$$= \left[\varphi^{\xi(+)}(s), \varphi^{\xi'(-)}(\tau) \right] = \frac{\delta_{\xi, \xi'}}{i} D^{(-)}(s - \tau), \quad (231)$$

(here $D^{(-)}(s) = \lim_{\varsigma \rightarrow \infty} \mathcal{D}^{(-)}(s, \varsigma; \mu_1)$, see Appendix B), but for both of them:

$$\langle 0\tilde{0} | \varphi^\xi(s; [\pm]\varsigma) \varphi^{\xi'}(\tau; [\pm]\varsigma) | 0\tilde{0} \rangle = [\varphi^{\xi(+)}(s; [\pm]\varsigma), \varphi^{\xi'(-)}(\tau; [\pm]\varsigma)], \quad (232)$$

$$\begin{aligned} [\varphi^{\xi(\pm)}(s; [\pm]\varsigma), \varphi^{\xi'(\mp)}(\tau; [\pm]\varsigma)] &= (\pm 1) \frac{\delta_{\xi, \xi'}}{i} \mathcal{D}^{(-)}(\pm(s - \tau), \varsigma; \mu_1) = \\ &= (\mp 1) \frac{1}{4\pi} \delta_{\xi, \xi'} \left\{ \ln \left(i \bar{\mu} \frac{\varsigma}{\pi} \sinh \left(\frac{\pi}{\varsigma} (\pm(s - \tau) - i0) \right) \right) - g(\varsigma, \mu_1) \right\}, \end{aligned} \quad (233)$$

$$\begin{aligned} [\tilde{\varphi}^{\xi(\pm)}(s; [\pm]\varsigma), \tilde{\varphi}^{\xi'(\mp)}(\tau; [\pm]\varsigma)] &= (\mp 1) \frac{\delta_{\xi, \xi'}}{i} \tilde{\mathcal{D}}^{(-)}(\pm(s - \tau), \varsigma; \mu_1) = \\ &= (\mp 1) \frac{1}{4\pi} \delta_{\xi, \xi'} \left\{ \ln \left(i \bar{\mu} \frac{\varsigma}{\pi} \sinh \left(\frac{\pi}{\varsigma} (\mp(s - \tau) - i0) \right) \right) - g(\varsigma, \mu_1) \right\}, \end{aligned} \quad (234)$$

$$[\varphi^{\xi(\pm)}(s; [\pm]\varsigma), \tilde{\varphi}^{\xi'(\mp)}(\tau; [\pm]\varsigma)] = (\pm 1)[\pm 1] \frac{1}{4\pi} \delta_{\xi, \xi'} \left\{ \ln \left(\cosh \left(\frac{\pi}{\varsigma} (s - \tau) \right) \right) - f(\varsigma, \mu_2) \right\}, \quad (235)$$

$$[\varphi^{\xi(\pm)}(s; [\pm]\varsigma), Q^{\xi'(\mp)}([\pm]\varsigma)] = \delta_{\xi, \xi'} \left[\frac{i}{4} - (\pm 1) \left(\frac{\hat{x}^0 - s}{2\varsigma} \right) \right], \quad (236)$$

$$[\tilde{\varphi}^{\xi(\pm)}(s; [\pm]\varsigma), \tilde{Q}^{\xi'(\mp)}([\pm]\varsigma)] = \delta_{\xi, \xi'} \left[-\frac{i}{4} - (\pm 1) \left(\frac{\hat{x}^0 - s}{2\varsigma} \right) \right], \quad (237)$$

$$[\varphi^{\xi(\pm)}(s; [\pm]\varsigma), \tilde{Q}^{\xi'(\mp)}([\pm]\varsigma)] = (\pm 1)[\pm 1] \delta_{\xi, \xi'} \left(\frac{\hat{x}^0 - s}{2\varsigma} \right) = [\tilde{\varphi}^{\xi(\pm)}(s; [\pm]\varsigma), Q^{\xi'(\mp)}([\pm]\varsigma)], \quad (238)$$

$$[Q^{\xi(\pm)}([\pm]\varsigma), Q^{\xi'(\mp)}([\pm]\varsigma)] = (\pm 1) a_1 \delta_{\xi, \xi'} = [\tilde{Q}^{\xi(\pm)}([\pm]\varsigma), \tilde{Q}^{\xi'(\mp)}([\pm]\varsigma)], \quad (239)$$

$$[Q^{\xi(\pm)}([\pm]\varsigma), \tilde{Q}^{\xi'(\mp)}([\pm]\varsigma)] = (\pm 1)[\mp 1] a_2 \delta_{\xi, \xi'} = [\tilde{Q}^{\xi(\pm)}([\pm]\varsigma), Q^{\xi'(\mp)}([\pm]\varsigma)]. \quad (240)$$

Here the following quantities are defined (see Appendix B and C):

$$g(\varsigma, \mu_1) = \int_{\mu_1}^{\infty} \frac{dk^1}{k^0} \frac{2}{e^{\varsigma k^0} - 1} \Rightarrow \frac{2}{\varsigma \mu_1} - \ln \left(\frac{2\pi}{\varsigma \bar{\mu}_1} \right), \quad \bar{\mu}_1 = \mu_1 e^{C_3} \rightarrow 0, \quad \lim_{\varsigma \rightarrow \infty} g(\varsigma, \mu_1) = 0, \quad (241)$$

$$f(\varsigma, \mu_2) = \int_{\mu_2}^{\infty} \frac{dk^1}{k^0} \frac{1}{\sinh(\varsigma k^0/2)} \Rightarrow \frac{2}{\varsigma \mu_2} - \ln 2, \quad \mu_2 \rightarrow 0, \quad \lim_{\varsigma \rightarrow \infty} f(\varsigma, \mu_2) = 0, \quad (242)$$

$$\delta_L(k^1) = \int_{-\infty}^{\infty} \frac{dx^1}{2\pi} \Delta \left(\frac{x^1}{L} \right) e^{\pm i k^1 x^1} \equiv L \bar{\Delta}(k^1 L), \quad L \rightarrow \infty, \quad \lim_{L \rightarrow \infty} \delta_L(k^1) = \delta(k^1), \quad (243)$$

$$a_0 = a_0(L) = \pi \int_0^{\infty} dk^1 k^1 \left(\delta_L(k^1) \right)^2 \Rightarrow \pi \int_0^{\infty} dt t \left(\bar{\Delta}(t) \right)^2 \equiv \pi I_1^{\Delta}, \quad I_n^{\Delta} \equiv \int_0^{\infty} dt t^n \left(\bar{\Delta}(t) \right)^2, \quad (244)$$

$$a_1 = a_1(L, \varsigma) = a_0(L) + 2\pi \int_0^{\infty} dk^1 k^1 \frac{(\delta_L(k^1))^2}{e^{\varsigma k^0} - 1} = a_0(L) + 2\pi \int_0^{\infty} dt t \frac{(\bar{\Delta}(t))^2}{e^{\varsigma t/L} - 1} \Rightarrow \quad (245)$$

$$\xRightarrow{L \rightarrow \infty} 2\pi I_0^{\Delta} \frac{L}{\varsigma} + \frac{\pi}{6} I_2^{\Delta} \frac{\varsigma}{L} + O \left(\left(\frac{\varsigma}{L} \right)^3 \right), \quad \lim_{\varsigma \rightarrow \infty} a_1(L, \varsigma) = a_0(L), \quad (246)$$

$$a_2 = a_2(L, \varsigma) = \pi \int_0^\infty dk^1 k^1 \frac{(\delta_L(k^1))^2}{\sinh(\varsigma k^0/2)} = \pi \int_0^\infty dt t \frac{(\overline{\Delta}(t))^2}{\sinh(t\varsigma/2L)} \Rightarrow \quad (247)$$

$$\xRightarrow{L \rightarrow \infty} 2\pi I_0^\Delta \frac{L}{\varsigma} + \left(\frac{\pi}{6} - \frac{\pi}{4}\right) I_2^\Delta \frac{\varsigma}{L} + O\left(\left(\frac{\varsigma}{L}\right)^3\right), \quad \lim_{\varsigma \rightarrow \infty} a_2(L, \varsigma) = 0, \quad (248)$$

where C_\Im is again the Euler-Mascheroni constant, and appearance of additional infrared regulators μ_1, μ_2 [51, 55] is clarified in Appendix B. It is worth to note that for chosen general type of volume cut-off regularization (31), (210) with arbitrary appropriate even function $\Delta(x^1/L)$ (243) the value of a_0 (43), (244), if it exists (is finite), does not depend on L at all, while the a_1 (245) and a_2 (247) in any case have the same divergent asymptotic behaviour (246), (248) for $L \rightarrow \infty$, but, in general, have different behaviour at $\varsigma \rightarrow \infty$. Here the existence of I_n^Δ for $n = 0, 1, 2$ is assumed, that for different regularizations are displayed in the Table of Appendix C. It is important to note that in any case the difference $a_1 - a_2$ becomes L -independent at $L \rightarrow \infty$, and if a_0 is finite, then $a_1 - a_2 \rightarrow 0$ at $L \rightarrow \infty$ (see Appendix C).

The $(\hat{x}^0 - s)$ -dependence of commutators (236)–(238) has no physical meaning and below will be eliminated automatically. But it seems convenient to retain it for additional control up to the end of calculation, because further it suggests the way of correct doubling of the number of degrees of freedom.

Following [33], by the use of the (pseudo) scalar fields given above, one can again construct a representation of solutions of the Dirac equation for a free massless trial field at finite temperature, $\partial_\xi \chi_\xi(x^{-\xi}, \varsigma) = 0$, in the form of local normal-ordered exponentials of the left and right bosonic thermofields $\varphi^{-\xi}(x^{-\xi}, \varsigma)$, and their charges $Q^\xi(\varsigma)$ (211), (212). The naive expression, which implies the bosonization relations (207) for the currents (202)–(204) with $Z_{(\chi)}(a) = 1$, is obtained from (55), (56), (71)–(77), (221) as:

$$\mathcal{V}_{\vartheta(B)}^{\mp 1} \chi_\xi(x^{-\xi}) \mathcal{V}_{\vartheta(B)}^{\pm 1} = \mathcal{N}_\varphi \left\{ \exp \left(-i2\sqrt{\pi} \left[\varphi^{-\xi}(x^{-\xi}; [\pm]\varsigma) + \frac{\xi}{4} Q^\xi([\pm]\varsigma) \right] \right) \right\} u_\xi(\mu_1, \varsigma), \quad (249)$$

$$u_\xi(\mu_1, \varsigma) = \left(\frac{\overline{\mu}}{2\pi} \right)^{1/2} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -\frac{g(\varsigma, \mu_1)}{2} \right\} \exp \left\{ -a_1(L) \frac{\pi}{8} \right\}, \quad (250)$$

where ϖ and Θ are again arbitrary initial overall and relative phases. It is worth to note that to have a correct zero temperature limit $\varsigma \rightarrow \infty$ for this field, here it is necessary to keep finite the values of all infrared regulators: $\overline{\mu}, \mu_1, L, a_1(L)$. However, the kinematic independence (193), (197), of the tilde-partners can be achieved now only by “admixing” the Klein factors of both the charges $\tilde{Q}^\xi(\varsigma)$ and $\tilde{Q}^{-\xi}(\varsigma)$ to the same field. Moreover, according to the meaning of L as macroscopic parameter explained above, the wanted thermofield should have a correct thermodynamic limit $L \rightarrow \infty$ for the finite temperature $T > 0$. To this end we define the new charges with simple commutation relations following from (236)–(240):

$$G^\xi([\pm]\varsigma) = Q^\xi([\pm]\varsigma) + [\pm 1] \tilde{Q}^\xi([\pm]\varsigma), \quad \tilde{G}^\xi([\pm]\varsigma) = \tilde{Q}^\xi([\pm]\varsigma) + [\pm 1] Q^\xi([\pm]\varsigma), \quad \text{with:} \quad (251)$$

$$\left[\varphi^{\xi(\pm)}(s; [\pm]\varsigma), G^{\xi'(\mp)}([\pm]\varsigma) \right] = \frac{i}{4} \delta_{\xi, \xi'}, \quad \left[\tilde{\varphi}^{\xi(\pm)}(s; [\pm]\varsigma), \tilde{G}^{\xi'(\mp)}([\pm]\varsigma) \right] = -\frac{i}{4} \delta_{\xi, \xi'}, \quad (252)$$

$$\left[\varphi^{\xi(\pm)}(s; [\pm]\varsigma), \tilde{G}^{\xi'(\mp)}([\pm]\varsigma) \right] = \frac{i}{4} [\pm 1] \delta_{\xi, \xi'}, \quad (253)$$

$$\left[G^{\xi(\pm)}([\pm]\varsigma), G^{\xi'(\mp)}([\pm]\varsigma) \right] = (\pm 1) 2(a_1 - a_2) \delta_{\xi, \xi'}, \quad (254)$$

$$\left[G^{\xi(\pm)}([\pm]\varsigma), \tilde{G}^{\xi'(\mp)}([\pm]\varsigma) \right] = (\pm 1) [\pm 1] 2(a_1 - a_2) \delta_{\xi, \xi'}. \quad (255)$$

Thus induced natural generalization, which, due to (27), again gives nevertheless the bosonization relations (207) for the currents (202)–(204) of the trial physical fields $\chi(x; [\pm]\varsigma)$ with $Z_{(\chi)}(a) = 1$, reads:

$$\chi_\xi(x^{-\xi}; [\pm]\varsigma) = \mathcal{N}_\varphi \left(\exp \left\{ R_\xi(x^{-\xi}; [\pm]\varsigma) \right\} \right) \hat{u}_\xi(\mu_1, [\pm]\varsigma), \quad (256)$$

$$R_\xi(x^{-\xi}; [\pm]\varsigma) = -i2\sqrt{\pi} \left[\varphi^{-\xi}(x^{-\xi}; [\pm]\varsigma) + \frac{1}{4}\sigma_0^\xi G^{-\xi}([\pm]\varsigma) + \frac{1}{4}\sigma_1^\xi G^\xi([\pm]\varsigma) \right], \quad (257)$$

$$\hat{u}_\xi(\mu_1, [\pm]\varsigma) = \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -\frac{g(\varsigma, \mu_1)}{2} \right\} \exp \left\{ -\frac{\pi}{4}(a_1 - a_2) \left[(\sigma_0^\xi)^2 + (\sigma_1^\xi)^2 \right] \right\}. \quad (258)$$

Independently the same expression for \hat{u}_ξ together with admissible values of $\sigma_{0,1}^\xi$ follows from the anti-commutation relations (8)–(10), kinematic independence conditions (193), (197), and symmetry under the above tilde - operation (174), [48], as:

$$\text{from } \{ \chi, \chi^\# \} : \frac{\sigma_1^\xi - \sigma_1^{-\xi}}{2} = \xi(2n_1 + 1), \quad \text{with arbitrary integer } n_0, n_1, n_2, n_3, \quad (259)$$

$$\text{from } \{ \chi, \tilde{\chi}^\# \} : \frac{\sigma_1^\xi + \sigma_1^{-\xi}}{2} = 2n_2 + 1, \quad \text{and: } \sigma_0^\xi = n_0 - n_3 + \xi(n_0 + n_3 + 1). \quad (260)$$

Contrary to (249) and [51], thus chosen free field is a simple transformation of its zero temperature case furnished by all necessary tilde Klein factors and connected with the solution (54)–(56) by the following steps. The first one is the generalization to the two-parametric family (154) of zero temperature solutions of free Dirac equation. Then, the second one is prompted by Eqs. (54), (55) as following.

By virtue of the relations (53), (71)–(77), (148), (154)–(156), (236)–(240), (248), (251), this suggests the simple admixing of the all necessary tilde Klein factors at zero temperature and directly leads to the field (256) for zero temperature ($\varsigma = \infty$):

$$\chi_\xi(x^{-\xi}; [\pm]\infty) = \chi_\xi(x^{-\xi}, \sigma, \rho) \exp \left[-i[\pm 1] \frac{\sqrt{\pi}}{2} \sigma_0^\xi \tilde{Q}^{-\xi} \right] \exp \left[-i[\pm 1] \frac{\sqrt{\pi}}{2} \sigma_1^\xi \tilde{Q}^\xi \right], \quad \text{or:} \quad (261)$$

$$\chi_\xi(x^{-\xi}; [\pm]\infty) = \mathcal{N}_\varphi \left(\exp \left\{ R_\xi(x^{-\xi}; [\pm]\infty) \right\} \right) \hat{u}_\xi([\pm]\infty), \quad \sigma_0^\xi = -\xi\sigma, \quad \sigma_1^\xi = \xi 1 + \rho, \quad (262)$$

$$R_\xi(x^{-\xi}; [\pm]\infty) = B_\xi(x^{-\xi}, \sigma, \rho) - i[\pm 1] \frac{\sqrt{\pi}}{2} \sigma_0^\xi \tilde{Q}^{-\xi} - i[\pm 1] \frac{\sqrt{\pi}}{2} \sigma_1^\xi \tilde{Q}^\xi, \quad (263)$$

$$\hat{u}_\xi([\pm]\infty) = \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -a_0 \frac{\pi}{4} \left[(\sigma_0^\xi)^2 + (\sigma_1^\xi)^2 \right] \right\} = \lim_{\varsigma \rightarrow \infty} \hat{u}_\xi(\mu_1, [\pm]\varsigma), \quad \text{so that:} \quad (264)$$

$$R_\xi(x^{-\xi}; [\pm]\varsigma) = \mathcal{V}_{\vartheta(B)}^{\mp 1} R_\xi(x^{-\xi}; [\pm]\infty) \mathcal{V}_{\vartheta(B)}^{\pm 1}, \quad \chi_\xi(x^{-\xi}; [\pm]\varsigma) = \mathcal{V}_{\vartheta(B)}^{\mp 1} \chi_\xi(x^{-\xi}; [\pm]\infty) \mathcal{V}_{\vartheta(B)}^{\pm 1}. \quad (265)$$

This means, that in accordance with the fermionic thermofield transformation, which due to Eq. (174), of course, gives automatically the field with all necessary anticommutation properties uniformly with continuous limit to zero temperature case (364), (365) of Appendix E:

$$\begin{aligned} \chi_\xi(x^{-\xi}; [\pm]\varsigma) &= \mathcal{V}_{\vartheta(F)}^{\mp 1} \chi_\xi(x^{-\xi}; [\pm]\infty) \mathcal{V}_{\vartheta(F)}^{\pm 1} = \\ &= \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{2\pi}} \left[\theta(\xi p^1) b(p^1; [\pm]\varsigma) e^{-i(p x)} + \xi \theta(\xi p^1) f^\dagger(p^1; [\pm]\varsigma) e^{i(p x)} \right] e^{i\varpi - i\xi\Theta/4}, \end{aligned} \quad (266)$$

the bosonic field representation (256)–(258) of fermi field also reproduce such commutation and continuity properties at $T \rightarrow 0$ due to the introduced Klein factors in Eqs. (261)–(263), that accords with the necessity of doubling the number of degrees of freedom already at $T = 0$ [35, 48]. From the conditions (148), (259), (260), (262), we conclude that it is enough without loss of generality to take for $\xi = \pm$:

$$n_3 = n_0 = \ell, \quad \sigma = -(2\ell + 1), \quad \sigma_0^\xi = -\xi\sigma; \quad n_1 = 0, \quad n_2 = n, \quad \rho = 2n + 1, \quad \sigma_1^\xi = \xi + \rho. \quad (267)$$

This induce inevitable additional ξ -dependence for c-number spinor $\hat{u}_\xi(\mu_1, [\pm]\varsigma)$ (258), if $a_1 - a_2 \neq 0$. Indeed, the last exponential factor in (258), if it differ from unity (as for the usual box, see (363) and the Table in Appendix C), leads to non-physical regularization dependent “temperature induced anomalous dimensions”, different for various components of this field.

Nevertheless, since due to (246), (248) and the discussion in Appendix C, excluding the case of usual box, for any continuous regularizations $\Delta(y^1/L)$: a_0 is finite and $a_1 - a_2 \Rightarrow 0$, for $L \rightarrow \infty$, then the c-number spinor (258) is reduced to its most simple “Oksak’s” form, which directly has a correct zero temperature behavior for Oksak case $a_0 = 0$ in (56), [33, 34] (see Appendix C):

$$\hat{u}_\xi(\mu_1, [\pm]\varsigma) \Rightarrow \left(\frac{\bar{\mu}}{2\pi}\right)^{1/2} e^{i\varpi - i\xi\Theta/4} \exp\left\{-\frac{g(\varsigma, \mu_1)}{2}\right\} = \hat{u}_\xi^{Ok}(\mu_1, [\pm]\varsigma), \quad \lim_{\varsigma \rightarrow \infty} \hat{u}_\xi^{Ok}(\mu_1, [\pm]\varsigma) = u_\xi^{Ok}. \quad (268)$$

3.6 Integration of the Heisenberg equation

For the chosen representation (207)–(258) the operator product in the linearized by means of (205), (206) HEq (194) or (195) is naturally redefined again into the normal-ordered form [33] with respect to the fields $\varphi^\xi(x^\xi, \varsigma)$:

$$\partial_0 \Psi_\xi(x, \varsigma) = \left(-\xi \partial_1 - i \frac{\beta g}{2\sqrt{\pi}} \hat{J}_{(x)}^{\xi(-)}(x, \varsigma)\right) \Psi_\xi(x, \varsigma) - \Psi_\xi(x, \varsigma) \left(i \frac{\beta g}{2\sqrt{\pi}} \hat{J}_{(x)}^{\xi(+)}(x, \varsigma)\right). \quad (269)$$

As above, the famous expression for the derivative of function $F(x^1)$ in terms of the operator P^1 : $-i\partial_1 F(x^1) = [P^1, F(x^1)]$ and its finite-shift equivalent: $e^{iaP^1} F(x^1) e^{-iaP^1} = F(x^1 + a)$ allows to transcribe the equation (269) for $x^0 = t$, $\Psi_\xi(x, \varsigma) \longleftrightarrow Y(t)$, as follows:

$$\frac{d}{dt} Y(t) = A(t)Y(t) - Y(t)B(t), \quad (270)$$

and to have again the following formal solution in terms of the time-ordered exponential:

$$Y(t) = T_A \left\{ \exp \left(\int_0^t d\tau A(\tau) \right) \right\} Y(0) \left[T_B \left\{ \exp \left(\int_0^t d\tau B(\tau) \right) \right\} \right]^{-1}, \quad (271)$$

which in this case is immediately transformed to the usual one for $Y(t) \rightarrow \Psi_\xi(x, \varsigma)$:

$$\Psi_\xi(x, \varsigma) = e^{C^{\xi(-)}(x, \varsigma)} \Psi_\xi(x^1 - \xi x^0, 0, \varsigma) e^{C^{\xi(+)}(x, \varsigma)}. \quad (272)$$

Using the operator bosonization (207) for the vector current of the trial physical field (256), we find as above:

$$\begin{aligned} C^{\xi(\pm)}(x, \varsigma) &= -i \frac{\beta g}{2\sqrt{\pi}} \int_0^{x^0} dy^0 \hat{J}_{(\chi)}^{-\xi(\pm)}(x^1 + \xi y^0 - \xi x^0, y^0, \varsigma) = \\ &= -i \frac{\beta g}{2\pi} \left[\varphi^{(\pm)}(x^1, x^0, \varsigma) - \varphi^{(\pm)}(x^1 - \xi x^0, 0, \varsigma) \right] = -i \frac{\beta g}{2\pi} \left[\varphi^{\xi(\pm)}(x^\xi, \varsigma) - \varphi^{\xi(\pm)}(-x^{-\xi}, \varsigma) \right]. \end{aligned} \quad (273)$$

Remarkably, that the completely unknown “initial” HF $\Psi_\xi(x^1 - \xi x^0, 0, \varsigma) = \lambda_\xi(x^{-\xi}, \varsigma)$ appears here also as a solution of free massless Dirac equation, $\partial_\xi \lambda_\xi(x^{-\xi}, \varsigma) = 0$, but certainly again unitarily inequivalent to the free field $\chi(x, \varsigma)$ (256). The expressions (272), (273) suggest to choose it also in the normal-ordered form with respect to the field φ , using appropriate “bosonic canonical transformation” of this field with parameters $\bar{\alpha} = 2\sqrt{\pi} \cosh \eta$, $\bar{\beta} = 2\sqrt{\pi} \sinh \eta$, connected by $\bar{\alpha}^2 - \bar{\beta}^2 = 4\pi$, which is generated now by the operator $F_\eta(\varsigma)$ (for $y^0 = x^0$) in the form $U_\eta(\varsigma) = \exp F_\eta(\varsigma)$, which as above (67), in fact does not depend on ξ and x^0 at all:

$$U_\eta^{-1}(\varsigma) \varphi(x, \varsigma) U_\eta(\varsigma) = \omega(x, \varsigma) \equiv \omega^\xi(x^\xi, \varsigma) + \omega^{-\xi}(x^{-\xi}, \varsigma) = \frac{1}{2\sqrt{\pi}} \left[\bar{\alpha} \varphi(x^1, x^0, \varsigma) + \bar{\beta} \varphi(x^1, -x^0, \varsigma) \right], \quad (274)$$

$$U_\eta^{-1}(\varsigma) \phi(x, \varsigma) U_\eta(\varsigma) = \Omega(x, \varsigma) \equiv \xi \left(\omega^{-\xi}(x^{-\xi}, \varsigma) - \omega^\xi(x^\xi, \varsigma) \right) = \frac{1}{2\sqrt{\pi}} \left[\bar{\alpha} \phi(x^1, x^0, \varsigma) - \bar{\beta} \phi(x^1, -x^0, \varsigma) \right], \quad (275)$$

$$U_\eta^{-1}(\varsigma) \varphi^\xi(x^\xi, \varsigma) U_\eta(\varsigma) = \omega^\xi(x^\xi, \varsigma) = \frac{1}{2\sqrt{\pi}} \left[\bar{\alpha} \varphi^\xi(x^\xi, \varsigma) + \bar{\beta} \varphi^{-\xi}(-x^\xi, \varsigma) \right] \equiv \mathcal{V}_{\vartheta(B)}^{\mp 1} \omega^\xi(x^\xi) \mathcal{V}_{\vartheta(B)}^{\pm 1}, \quad (276)$$

$$U_\eta^{-1}(\varsigma) Q^\xi([\pm]\varsigma) U_\eta(\varsigma) = \mathcal{W}^\xi([\pm]\varsigma) = \frac{1}{2\sqrt{\pi}} \left[\bar{\alpha} Q^\xi([\pm]\varsigma) - \bar{\beta} Q^{-\xi}([\pm]\varsigma) \right] \equiv \mathcal{V}_{\vartheta(B)}^{\mp 1} \mathcal{W}^\xi \mathcal{V}_{\vartheta(B)}^{\pm 1}, \quad (277)$$

$$U_\eta^{-1}(\varsigma) G^\xi([\pm]\varsigma) U_\eta(\varsigma) = \mathcal{G}^\xi([\pm]\varsigma) = \frac{1}{2\sqrt{\pi}} \left[\bar{\alpha} G^\xi([\pm]\varsigma) - \bar{\beta} G^{-\xi}([\pm]\varsigma) \right] = \mathcal{W}^\xi([\pm]\varsigma) + [\pm 1] \tilde{\mathcal{W}}^\xi([\pm]\varsigma), \quad (278)$$

$$F_\eta([\pm]\varsigma) = \mathcal{V}_{\vartheta(B)}^{\mp 1} \mathbf{F}_\eta[c(k^1)] \mathcal{V}_{\vartheta(B)}^{\pm 1} + \mathcal{V}_{\vartheta(B)}^{\mp 1} \tilde{\mathbf{F}}_\eta[\tilde{c}(k^1)] \mathcal{V}_{\vartheta(B)}^{\pm 1} = \mathbf{F}_\eta \left[c(k^1; [\pm]\varsigma) \right] + \tilde{\mathbf{F}}_\eta \left[\tilde{c}(k^1; [\pm]\varsigma) \right] \equiv \quad (279)$$

$$\begin{aligned} &\equiv \eta \int_{-\infty}^{\infty} \frac{dk^1 \theta(k^1)}{2\pi 2k^0} \left[c(k^1; [\pm]\varsigma) c(-k^1; [\pm]\varsigma) - c^\dagger(k^1; [\pm]\varsigma) c^\dagger(-k^1; [\pm]\varsigma) \right] + \\ &+ \eta \int_{-\infty}^{\infty} \frac{dk^1 \theta(k^1)}{2\pi 2k^0} \left[\tilde{c}(k^1; [\pm]\varsigma) \tilde{c}(-k^1; [\pm]\varsigma) - \tilde{c}^\dagger(k^1; [\pm]\varsigma) \tilde{c}^\dagger(-k^1; [\pm]\varsigma) \right], \quad \text{or:} \end{aligned} \quad (280)$$

$$F_\eta(\varsigma) = 2i\eta \int_{-\infty}^{\infty} dy^1 \varphi^\xi(y^\xi, \varsigma) \partial_0 \varphi^{-\xi}(-y^\xi, \varsigma) - 2i\eta \int_{-\infty}^{\infty} dy^1 \tilde{\varphi}^\xi(y^\xi, \varsigma) \partial_0 \tilde{\varphi}^{-\xi}(-y^\xi, \varsigma) = \quad (281)$$

$$= 2i\eta \int_{-\infty}^{\infty} dy^1 \omega^\xi(y^\xi, \varsigma) \partial_0 \omega^{-\xi}(-y^\xi, \varsigma) - 2i\eta \int_{-\infty}^{\infty} dy^1 \tilde{\omega}^\xi(y^\xi, \varsigma) \partial_0 \tilde{\omega}^{-\xi}(-y^\xi, \varsigma), \quad \text{with:} \quad (282)$$

$$\left[\varphi^{\xi(\pm)}(s, \varsigma), F_\eta(\varsigma) \right] = \eta \varphi^{-\xi(\mp)}(-s, \varsigma), \quad \left[Q^{\xi(\pm)}(\hat{x}^0, \varsigma), F_\eta(\varsigma) \right] = -\eta Q^{-\xi(\mp)}(-\hat{x}^0, \varsigma), \quad \hat{x}^0 \Rightarrow 0. \quad (283)$$

As shown in Appendix D, here the second tilde-conjugate term in formulas (279)–(282) is very important, providing $\tilde{F}_\eta(\varsigma) = F_\eta(\varsigma)$. For the field $\lambda_\xi(x^{-\xi}, \varsigma)$ by the same way as above (71)–(82), keeping in mind the relations (126), (146)–(148), one obtains, with: $\Sigma_0^\xi = \bar{\alpha}\sigma_0^\xi - \bar{\beta}\sigma_1^\xi$, $\Sigma_1^\xi = \bar{\alpha}\sigma_1^\xi - \bar{\beta}\sigma_0^\xi$, that:

$$\lambda_\xi(x^{-\xi}, \varsigma) = U_\eta^{-1}(\varsigma)\chi_\xi(x^{-\xi}; \varsigma)U_\eta(\varsigma) = \lambda_\xi(x^{-\xi}; [\pm]\varsigma) = \mathcal{N}_\varphi \left(\exp \left\{ \mathcal{R}_\xi(x^{-\xi}; [\pm]\varsigma) \right\} \right) w_\xi(\mu_1, \varsigma), \quad (284)$$

$$\mathcal{R}_\xi(x^{-\xi}; [\pm]\varsigma) = -i2\sqrt{\pi} \left[\omega^{-\xi}(x^{-\xi}; [\pm]\varsigma) + \frac{\sigma_0^\xi}{4}\mathcal{G}^{-\xi}([\pm]\varsigma) + \frac{\sigma_1^\xi}{4}\mathcal{G}^\xi([\pm]\varsigma) \right], \quad \text{or:} \quad (285)$$

$$\mathcal{R}_\xi(x^{-\xi}; [\pm]\varsigma) = -i \left[2\sqrt{\pi}\omega^{-\xi}(x^{-\xi}; [\pm]\varsigma) + \frac{\Sigma_0^\xi}{4}\mathcal{G}^{-\xi}([\pm]\varsigma) + \frac{\Sigma_1^\xi}{4}\mathcal{G}^\xi([\pm]\varsigma) \right], \quad (286)$$

$$w_\xi(\mu_1, \varsigma) = \left(\frac{\bar{\mu}}{2\pi} \right)^{1/2} \left(\frac{\bar{\mu}}{\Lambda} \right)^{\bar{\beta}^2/4\pi} e^{i\varpi - i\xi\Theta/4} \exp \left\{ -g(\varsigma, \mu_1) \left(\frac{1}{2} + \frac{\bar{\beta}^2}{4\pi} \right) \right\} \\ \cdot \exp \left\{ -(a_1 - a_2) \frac{\pi}{4} \left([(\sigma_0^\xi)^2 + (\sigma_1^\xi)^2] \cosh 2\eta - 2\sigma_0^\xi\sigma_1^\xi \sinh 2\eta \right) \right\}. \quad (287)$$

The CAR (94) may be easily verified for this field with $Z_{(\lambda)}(a)$ (83). For the corresponding current $\hat{J}_{(\lambda)}^\mu(x, \varsigma)$, defined by Eqs. (202)–(204), or by the Johnson definition [2, 3, 7], with the same zero temperature renormalization constant $Z_{(\lambda)}(a)$ (83), one finds again the previous bosonization rules (207):

$$\hat{J}_{(\lambda)}^\mu(x, \varsigma) = \frac{1}{\sqrt{\pi}} \partial^\mu \omega(x, \varsigma) = -\frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \Omega(x, \varsigma), \quad (288)$$

onto the new scalar fields $\omega(x, \varsigma)$, $\Omega(x, \varsigma)$, $\omega^\xi(x^\xi, \varsigma)$, $\mathcal{W}^\xi([\pm]\varsigma)$, obey the same commutation relations, as initial fields $\varphi^\xi(x^\xi, \varsigma)$ (213)–(215), (219):

$$[\omega^\xi(s, \varsigma), \omega^{\xi'}(\tau, \varsigma)] = -\frac{i}{4} \varepsilon(s - \tau) \delta_{\xi, \xi'}, \quad [\omega^\xi(s, \varsigma), \mathcal{W}^{\xi'}(\varsigma)] = \frac{i}{2} \delta_{\xi, \xi'}, \quad (289)$$

$$[\omega^\xi(s, \varsigma), \tilde{\omega}^{\xi'}(\tau, \varsigma)] = 0, \quad [\omega^\xi(s, \varsigma), \tilde{\mathcal{W}}^{\xi'}(\varsigma)] = 0, \quad [\mathcal{W}^\xi(\varsigma), \tilde{\mathcal{W}}^{\xi'}(\varsigma)] = 0. \quad (290)$$

Substituting the normal form (284) into the solution (272), by imposing the same conditions (89) onto the parameters $\bar{\alpha}, \bar{\beta}$, that are necessary to have correct Lorentz-transformation properties corresponding to the spin 1/2, and correct canonical anticommutation relations, we again immediately obtain the normal exponential of the DM for Thirring field in the form, analogous to Oksak solution (88), [33, 34], where the condition $\bar{\beta} = \beta g/(2\pi)$ from (89) again replaces the thermofield $\omega^{-\xi}(x^{-\xi}, \varsigma)$ by thermofield $\varrho^{-\xi}(x; \varsigma)$, and the use of the charges (251)–(255) assures a correct doubling of the number of degrees of freedom:

$$\Psi_\xi(x; [\pm]\varsigma) = \mathcal{N}_\varphi \left(\exp \left\{ \mathfrak{R}_\xi(x; [\pm]\varsigma) \right\} \right) w_\xi(\mu_1, \varsigma), \quad (291)$$

$$\mathfrak{R}_\xi(x; [\pm]\varsigma) = -i2\sqrt{\pi} \left[\varrho^{-\xi}(x; [\pm]\varsigma) + \frac{\sigma_0^\xi}{4}\mathcal{G}^{-\xi}([\pm]\varsigma) + \frac{\sigma_1^\xi}{4}\mathcal{G}^\xi([\pm]\varsigma) \right], \quad \text{or:} \quad (292)$$

$$\mathfrak{R}_\xi(x; [\pm]\varsigma) = -i \left[2\sqrt{\pi} \varrho^{-\xi}(x; [\pm]\varsigma) + \frac{\Sigma_0^\xi}{4}\mathcal{G}^{-\xi}([\pm]\varsigma) + \frac{\Sigma_1^\xi}{4}\mathcal{G}^\xi([\pm]\varsigma) \right], \quad (293)$$

$$2\sqrt{\pi} \varrho^{-\xi}(x; [\pm]\varsigma) = \overline{\alpha}\varphi^{-\xi}(x^{-\xi}; [\pm]\varsigma) + \overline{\beta}\varphi^{\xi}(x^{\xi}; [\pm]\varsigma), \quad (294)$$

$$\text{with: } \Sigma_0^{\xi} = \overline{\alpha}\sigma_0^{\xi} - \overline{\beta}\sigma_1^{\xi}, \quad \Sigma_1^{\xi} = \overline{\alpha}\sigma_1^{\xi} - \overline{\beta}\sigma_0^{\xi}, \quad (295)$$

$$\text{and: } \sigma_0^{\xi} = -\xi\sigma \Rightarrow \xi(2\ell + 1), \quad \sigma_1^{\xi} = \xi 1 + \rho \Rightarrow \xi 1 + (2n + 1). \quad (296)$$

Straightforward calculation of the current operators (202)–(204) with the same current's (92) and field's (96) renormalization constant $Z_{(\Psi)}(a)$, arisen from the same short-distance behavior of Wightman functions (335), reproduces the CAR (94) and bosonization relations (205)–(207) as following weak equalities:

$$\widehat{J}_{(\Psi)}^{\nu}(x, \varsigma) \stackrel{w}{=} -\frac{\beta}{2\pi} \epsilon^{\mu\nu} \partial_{\nu} \phi(x, \varsigma) = \frac{\beta}{2\sqrt{\pi}} \widehat{J}_{(\chi)}^{\nu}(x, \varsigma), \quad \text{for: } Z_{(\chi)}(a) = 1, \quad Z_{(\Psi)}(a) = (-\Lambda^2 a^2)^{-\overline{\beta}^2/4\pi}, \quad (297)$$

where β is again defined by Eqs. (93). And vice versa, this bosonization relation and CAR (94) separately imply, similarly to the free case (256)–(258), the one and the same expression (287) for the c -number spinor $w_{\xi}(\mu_1, \varsigma)$ in terms of arbitrary parameters σ, ρ and $a_{1,2}$, with the same $Z_{(\Psi)}(a)$ (92), (297).

Remarkably, that the CAR (8), (94), jointly with locality (9), (10) and kinematic independence conditions (197) give for the fermionic fields $\chi(x, [\pm]\varsigma)$, $\lambda(x, [\pm]\varsigma)$, $\Psi(x, [\pm]\varsigma)$ separately the above one and the same relations (259), (260) for the parameters $\sigma_{0,1}^{\xi}$ simultaneously, leading to their final values (296).

The same calculation as above leads to the same values of the Johnson commutators (104)–(109) and the corresponding charge algebras (111)–(113).

For the non-mixed VEV of the strings of these fields (291), following to (127)–(129), by means of the formulae (326)–(330) from Appendix B, one can obtain, again denoting here $l_i \mathcal{R}_{\xi_i}(x_i; [\pm]\varsigma) \mapsto \mathcal{R}_i$:

$$\begin{aligned} & \left(\Lambda^{\overline{\beta}^2/4\pi} \sqrt{2\pi} \right)^p \left\langle 0 \left| \prod_{i=1}^p \Psi_{\xi_i}^{(l_i)}(x_i; [\pm]\varsigma) \right| 0 \right\rangle = \left\langle 0 \left| \mathcal{N}_{\varphi} \left\{ \exp \left(\sum_{j=1}^p \mathcal{R}_j \right) \right\} \right| 0 \right\rangle \exp \left\{ i\varpi \sum_{i=1}^p l_i - i\frac{\Theta}{4} \sum_{i=1}^p l_i \xi_i \right\} \\ & \cdot \left(\overline{\mu} \exp \left\{ -g(\varsigma, \mu_1) - (a_1 - a_2) \frac{\pi}{2} \left[(1 + \sigma)^2 + (\beta^2/4\pi)^2 \rho^2 \right] \right\} \right)^{\left(\sum_{i=1}^p l_i \right)^2 (\pi/\beta^2)} \\ & \cdot \left(\overline{\mu} \exp \left\{ -g(\varsigma, \mu_1) - (a_1 - a_2) \frac{\pi}{2} \left[(1 - \sigma)^2 + (4\pi/\beta^2)^2 \rho^2 \right] \right\} \right)^{\left(\sum_{i=1}^p l_i \xi_i \right)^2 (\beta^2/16\pi)} \\ & \cdot \exp \left\{ -(a_1 - a_2) \frac{\pi}{2} \rho [\cosh 2\eta + \sigma \sinh 2\eta] \left(\sum_{i=1}^p l_i \xi_i \right) \left(\sum_{j=1}^p l_j \right) \right\} \\ & \cdot \prod_{i < k}^p \left\{ e^{i\pi(\xi_i - \xi_k)} \left[\frac{\sinh \left(\frac{\pi}{\varsigma} (x_i^- - x_k^- - i0) \right)}{\sinh \left(\frac{\pi}{\varsigma} (x_i^+ - x_k^+ - i0) \right)} \right]^{\xi_i + \xi_k} \right. \\ & \cdot \left. \left[\left(\frac{i\varsigma}{\pi} \right)^2 \sinh \left(\frac{\pi}{\varsigma} (x_i^- - x_k^- - i0) \right) \sinh \left(\frac{\pi}{\varsigma} (x_i^+ - x_k^+ - i0) \right) \right]^{(4\pi/\beta^2) + \xi_i \xi_k (\beta^2/4\pi)} \right\}^{l_i l_k / 4}. \end{aligned} \quad (298)$$

The changing of the factor $a_0\pi/4$ to $(a_1 - a_2)\pi/2$ on comparing with (157) corresponds to doubling of the number of degree of freedom. Only under both superselection rules (131), (132) the expression (298)

simplifies to:

$$\begin{aligned}
& \left(\Lambda^{\bar{\beta}^2/4\pi\sqrt{2\pi}} \right)^p \left\langle 0 \left| \prod_{i=1}^p \Psi_{\xi_i}^{(l_i)}(x_i; [\pm]\varsigma) \right| 0 \right\rangle = \delta_{\sum_{i=1}^p l_i, 0} \delta_{\sum_{i=1}^p l_i \xi_i, 0} \\
& \cdot \prod_{i < k}^p \left\{ e^{i\pi(\xi_i - \xi_k)} \left[\frac{\sinh\left(\frac{\pi}{\varsigma}(x_i^- - x_k^- - i0)\right)}{\sinh\left(\frac{\pi}{\varsigma}(x_i^+ - x_k^+ - i0)\right)} \right]^{\xi_i + \xi_k} \right. \\
& \cdot \left. \left[\left(\frac{i\varsigma}{\pi}\right)^2 \sinh\left(\frac{\pi}{\varsigma}(x_i^- - x_k^- - i0)\right) \sinh\left(\frac{\pi}{\varsigma}(x_i^+ - x_k^+ - i0)\right) \right]^{(4\pi/\beta^2) + \xi_i \xi_k (\beta^2/4\pi)} \right\}^{l_i l_k / 4}. \quad (299)
\end{aligned}$$

Thus, in accordance with [51], only both zero temperature superselection rules (131), (132) assure again the elimination of all the old and new infrared divergences, regularized by parameters $\bar{\mu}$, L , $\bar{\mu}_1$, $a_{1,2}$, and elimination of all dependencies on parameters σ , ρ , ϖ , Θ , so that only the last two lines of Eqs. (298), or (299) survive again independently of the volume cut-off regularization function from the Table of Appendix C, and ultraviolet renormalization stays only necessary again. The zero temperature limit of Eq. (299) evidently gives the last lines of expressions (157), or (130), independently on type of the volume cut-off regularization was used. The mixed VEV of the HF with their tilde-partner reveal the similar properties.

The Eqs. (82), (287) easy show, that, for $a_1 - a_2 \Rightarrow 0$, with $L \rightarrow \infty$, e.g., for all continuous regularizations, both the old and new infrared divergences have one and the same character, given, as for zero temperature case, by the dynamical dimension $d_{(\Psi)}$ (114), (126), and again, as for the above free case (268), recasts the c- number spinor $w_\xi(\mu_1, \varsigma)$ (287) into its most simple ‘‘Oksak’s’’ form, which has a correct zero temperature behavior for Oksak case $a_0 = 0$ in Eq. (82):

$$w_\xi(\mu_1, \varsigma) \xrightarrow{L \rightarrow \infty} \left(\frac{\Lambda}{2\pi} \right)^{1/2} e^{i\varpi - i\xi\Theta/4} \exp \left\{ \left(\frac{1}{2} + \frac{\bar{\beta}^2}{4\pi} \right) \left[\ln \left(\frac{\bar{\mu}}{\Lambda} \right) - g(\varsigma, \mu_1) \right] \right\} = w_\xi^{Ok}(\mu_1, \varsigma), \quad (300)$$

$$\lim_{\varsigma \rightarrow \infty} w_\xi^{Ok}(\mu_1, \varsigma) = v_\xi^{Ok} \equiv v_\xi|_{a_0=0}. \quad (301)$$

However for the usual box (see Appendix C) the spinor $w_\xi(\mu_1, \varsigma)$, as well as the free one (258), due to the last exponential factor in (287) furnished by the relation (363), also acquires additional multiplier of ‘‘temperature induced anomalous dimension’’. The multiplicatively renormalized field may be achieved now only for the field of Morchio et al. (124), (150) with discrete values of coupling constant, which follows from (267), and recasts the last exponential $e^{\{\dots\}}$ of Eq. (287) to the common multiplier:

$$-\sigma = (2\ell + 1) \Rightarrow \coth 2\eta, \quad \frac{g}{\pi} = \sqrt{1 + \frac{1}{\ell}} - 1, \quad \text{whence: } e^{\{\dots\}} \mapsto \left(\frac{\pi}{\varsigma\Lambda} \right)^{\mathcal{M}(\ell, n)}, \quad (302)$$

$$\text{with: } \mathcal{M}(\ell, n) = \frac{(2\ell + 1)}{16\sqrt{\ell(\ell + 1)}} \left[(2\ell + 1)^2 + (2n + 1)^2 - 1 \right]. \quad (303)$$

As was discussed above, such a possibility of multiplicative renormalization is absent for the case of free field (256)–(258), which formally corresponds here to limit $\ell \rightarrow \infty$. Thus, so obtained non-perturbative solution looks as artifact of the use of the usual box for the charge regularization [50].

4 Conclusion

The main lesson of our work is very simple: the correct true HF should be only a fully normal ordered operator in the sense of DM onto irreducible physical fields. Only this form clarifies and assures correct renormalization, commutation and symmetry properties. It allows also a simple connections between different types of solutions with finite and zero temperature.

Contrary to the recent works [13]-[24],[50], [51]-[53], we consider different types of charge's regularization and take into account all possible mutual commutation relations of bosonic thermofields and their charges. We reveal that their non-physical $(\hat{x}^0 - s)$ - dependence fixes the correct doubling of the number of degrees of freedom, and thus self-consistently eliminates from the normal forms of the free and Heisenberg fields operators and from the VEV of their products.

The canonical transformations we found, mutually connect the different solutions by two additional parameters, which being arbitrary continuous for zero temperature solution, acquire only discrete nonzero values for the finite temperature solutions. The conditions (296) obtained for these parameters provide the anticommutation, locality and kinematic independence relations for both the free and Thirring fields and their tilde partners simultaneously.

We show that integration of HEqs by means of the linearization procedure and dynamical mapping onto the Schrödinger fields, - with generalized initial condition at $t = x^0 = 0$, $\lim_{t \rightarrow 0} \Psi(x^1, t) \stackrel{w}{=} \Upsilon[\psi_s(x^1, 0)]$, is relevant also for the finite temperature case. The observed weak linearization (23), (205) of HEqs with so-generalized initial conditions in a weak sense allows to overcome the restrictions of Haag theorem, removing them into the representation construction of Schrödinger physical fields, at first, as reducible massless free Dirac fields: $\chi(x)$, $\chi(x, \varsigma)$, and then, as irreducible massless (pseudo) scalar fields: $\phi(x)$, $\phi(x, \varsigma)$. The latter ones arisen as Schrödinger physical fields, in fact play the role of asymptotic ones. Due to automatical elimination of zero mode's contributions, the chosen here representation space of free massless pseudoscalar field relaxes the problem of non-positivity of its inner product induced by Wightman functions.

Within the thermofield dynamics formalism [35, 48] it is shown that for thus exactly linearizable and exactly solvable Thirring model at finite temperature the bosonization relations retain their operator sense at finite temperature only among the free fields operators. For the Heisenberg currents these bosonization rules are applicable only in a weak sense.

The general solutions for HF (61), (272) keeps the Klaiber's normal form [5], but with distinct unitarily inequivalent representation of the free massless Dirac field sandwiched the simple dynamical factors and generated by distinct unitarily inequivalent representation of free massless (pseudo) scalar field. The zero temperature limit of thermal solution gives two-parametric generalization of the known Oksak solution [33, 34].

The notion of "hot" and "cold" thermofields is found to be convenient to distinguish different thermofield representations giving the correct normal form of thermofield solution for finite temperature Thirring model with respect to different vacua. The new vacua always appear as infinite products of coherent states with respect to initial vacua of elementary field's oscillators.

We show that very popular volume cut-off regularization by usual box of length L , [50], which is crucial for the use of Bethe Ansatz method [28, 29], when it is used for the charge definition, leads to non-physical properties of temperature solutions. While, for any continuous charge regularization function excluding the usual box, we found one and the same consistent properties of the field solutions for both zero and the

finite temperature cases. Moreover, the latter case is independent of any type of continuous regularization at the corresponding thermodynamic limit $L \rightarrow \infty$. The non-mixed n - point's VEV are independent of any regularization at all, as well as on any non-physical parameters, if and only if the both superselection rules are fulfilled. And only for this case the thermodynamic limit $L \rightarrow \infty$ and the zero temperature limit $\varsigma \rightarrow \infty$ may be successfully interchanged for these Wightman functions with the one and the same result.

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5 Appendix

5.1 Appendix A

For arbitrary numbers λ, u, v , the following generalization of the results of [64] may be obtained by means of the calculation of the values $(u\partial g/\partial u \pm v\partial g/\partial v) g^{-1}$, for $g = g(u, v)$, with closed algebra of operators:

$$[A, B] = 2\lambda C, \quad [C, A] = A, \quad [C, B] = -B, \quad g(u, v) = \exp(uA + vB) : \quad (304)$$

$$g(u, v) = \exp \left\{ \sqrt{\frac{u}{\lambda v}} \tanh(\sqrt{\lambda uv}) A \right\} \exp \left\{ -2 \ln(\cosh(\sqrt{\lambda uv})) C \right\} \exp \left\{ \sqrt{\frac{v}{\lambda u}} \tanh(\sqrt{\lambda uv}) B \right\}, \quad (305)$$

where the main branches of analytic functions \sqrt{z} and $\ln z$ are assumed. By using this formula, the operator U_η^{-1} (66), (69) of Bogoliubov transformation transcribes with the help of operators $\mathcal{K}_{\pm,0}$ and/or $K_{\pm,0}(k^1)$, forming the algebras of the “big” and “small” groups $SU(1, 1)$ (here the length of **usual box** is L):

$$\mathcal{K}_\pm = \int_{-\infty}^{\infty} \frac{dk^1 \theta(k^1)}{2\pi 2k^0} \left\{ \begin{array}{c} c^\dagger(k^1) c^\dagger(-k^1) \\ c(-k^1) c(k^1) \end{array} \right\} \equiv \frac{L}{2\pi} \int_{-\infty}^{\infty} dk^1 K_\pm(k^1) \stackrel{L \rightarrow \infty}{\Longleftarrow} \sum_{k_n^1 = -\infty}^{\infty} K_\pm(k_n^1), \quad k_n^1 = \frac{2\pi n}{L}, \quad (306)$$

$$\mathcal{K}_0 = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dk^1 \theta(k^1)}{2\pi 2k^0} (c^\dagger(k^1) c(k^1) + c(-k^1) c^\dagger(-k^1)) \equiv \frac{L}{2\pi} \int_{-\infty}^{\infty} dk^1 K_0(k^1) \stackrel{L \rightarrow \infty}{\Longleftarrow} \sum_{k_n^1 = -\infty}^{\infty} K_0(k_n^1) = \quad (307)$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} dk^1 \theta(k^1) \left(\frac{c^\dagger(k^1) c(k^1) + c^\dagger(-k^1) c(-k^1)}{2\pi 2k^0} + \delta(0) \right), \quad \delta(k^1 - q^1) \Rightarrow \frac{L}{2\pi} \delta_{k^1, q^1}, \quad \delta(0) \Rightarrow \frac{L}{2\pi}, \quad (308)$$

$$[K_-(k^1), K_+(q^1)] = 2K_0(k^1) \delta_{k^1, q^1}, \quad [K_0(k^1), K_\pm(q^1)] = \pm K_\pm(k^1) \delta_{k^1, q^1}, \quad K_-(k^1) = \{K_+(k^1)\}^\dagger, \quad (309)$$

$$[\mathcal{K}_-, \mathcal{K}_+] = 2\mathcal{K}_0, \quad [\mathcal{K}_0, \mathcal{K}_\pm] = \pm \mathcal{K}_\pm, \quad \mathcal{K}_- = \{\mathcal{K}_+\}^\dagger, \quad \mathcal{C}_2 = \mathcal{K}_0^2 - \mathcal{K}_0 - \mathcal{K}_+ \mathcal{K}_- \Rightarrow \ell(\ell - 1) \hat{I}, \quad \text{as:} \quad (310)$$

$$U_\eta^{-1} = \exp\{-F_\eta\} \equiv \exp\{\eta[\mathcal{K}_+ - \mathcal{K}_-]\} \stackrel{L \rightarrow \infty}{\Longleftarrow} \prod_{k_n^1 = -\infty}^{\infty} \exp\left\{\eta[K_+(k_n^1) - K_-(k_n^1)]\right\}, \quad \text{for:} \quad (311)$$

$$|0\rangle \stackrel{L \rightarrow \infty}{\Longleftarrow} \prod_{k_n^1 = -\infty}^{\infty} |0_{k_n^1}\rangle, \quad \text{or:} \quad U_\eta^{-1} = \exp\{\tanh \eta \mathcal{K}_+\} \exp\{-2 \ln(\cosh \eta) \mathcal{K}_0\} \exp\{-\tanh \eta \mathcal{K}_-\}. \quad (312)$$

With: $\hat{n}(k^1) = \frac{c^\dagger(k^1)c(k^1)}{2k^0}$, $K_0(k^1) = \frac{\theta(k^1)}{2} \left[1 + \frac{\hat{n}(k^1) + \hat{n}(-k^1)}{L} \right]$, the Casimir operator are: (313)

$$C_2(k^1) = K_0^2(k^1) - K_0(k^1) - K_+(k^1)K_-(k^1) = \frac{\theta(k^1)}{4} \left[-1 + \left(\frac{\hat{n}(k^1) - \hat{n}(-k^1)}{L} \right)^2 \right] \Rightarrow \kappa(\kappa - 1)\hat{I}. \quad (314)$$

Here $\theta(k^1)\hat{n}(\pm k^1)/L$ are the density operators of right and left moving pseudoscalar particles with momentum k^1 , \hat{I} - unite operator. As a representation states of the “small” and “big” groups, the initial vacua for every oscillation mode k_n^1 , $c(k_n^1)|0_{k_n^1}\rangle = 0$ and the total vacuum (312) have the quantum numbers κ , $\nu = \kappa + m$, m - integer, and $\ell, N = \ell + m$, correspondingly, where for the “small” group:

$$C_2(k^1)|\kappa, \nu\rangle = \kappa(\kappa - 1)|\kappa, \nu\rangle, \quad K_0(k^1)|\kappa, \nu\rangle = \nu|\kappa, \nu\rangle, \quad |0_{k^1}\rangle \Rightarrow \left| \frac{1}{2}, \frac{1}{2} \right\rangle, \quad \nu \Rightarrow \kappa = \frac{1}{2}, \quad (315)$$

$$\text{whereas for the “big” one: } C_2|\ell, N\rangle = \ell(\ell - 1)|\ell, N\rangle, \quad K_0|\ell, N\rangle = N|\ell, N\rangle, \quad |0\rangle \Rightarrow |\ell, \ell\rangle, \quad (316)$$

$$\text{for: } N \Rightarrow \ell = \frac{L}{4\pi\rho}, \quad \text{with: } \frac{1}{2\pi\rho} = \frac{\Lambda}{2\pi} = \int_0^\Lambda \frac{dk^1}{2\pi} = \int_0^\infty \frac{dk^1}{2\pi} e^{-k^1/\Lambda}, \quad \text{so that: } |0\rangle \xleftarrow{\ell \rightarrow \infty} \prod_{n=-2\ell}^{2\ell} |0_{k_n^1}\rangle, \quad (317)$$

$$2\pi\rho - \text{ is the effective excitation volume for one mode, and } 2\ell = \frac{L}{2\pi\rho} \text{ is a number of excitations} \quad (318)$$

that may be inserted in the system volume L without overlap. The total new vacuum state $|\hat{0}\rangle$ (98) is obviously a coherent state (311), (312), [62] for the discrete series representation of “big” group $SU(1, 1)$ over the initial total vacuum $|0\rangle$, as infinite product (311), (317) of coherent states over one mode vacua $|0_{k_n^1}\rangle$ for the “small” groups for $\ell \rightarrow \infty$:

$$|\hat{0}\rangle = U_\eta^{-1}|0\rangle \longleftarrow (\cosh \eta)^{-2\ell} \exp \{ \tanh \eta \mathcal{K}_+ \} |0\rangle, \quad (319)$$

$$|\hat{0}\rangle = \left(1 - \tanh^2 \eta \right)^\ell \sum_{m=0}^\infty \left[\frac{\Gamma(m + 2\ell)}{m! \Gamma(2\ell)} \right]^{1/2} \tanh^m \eta |\ell, \ell + m\rangle, \quad \ell \rightarrow \infty, \quad (320)$$

$$|\ell, \ell + m\rangle = \left[\frac{\Gamma(2\ell)}{m! \Gamma(m + 2\ell)} \right]^{1/2} (\mathcal{K}_+)^m |\ell, \ell\rangle, \quad (321)$$

$$\langle 0|\hat{0}\rangle = \langle 0|U_\eta^{-1}|0\rangle = (\cosh \eta)^{-2\ell} = \left(1 - \tanh^2 \eta \right)^\ell \xrightarrow{\ell \rightarrow \infty} 0, \quad \text{for } \eta \neq 0, \quad (322)$$

what means the orthogonality of these states for $\ell \rightarrow \infty$ and unitary inequivalence of corresponding representations. For the thermal Bogoliubov transformation (184), (337) the corresponding value reads:

$$\langle 0\tilde{0}|0(\varsigma)\rangle = \langle 0\tilde{0}|\mathcal{V}_{\vartheta(B)}^{-1}|0\tilde{0}\rangle = \exp \left\{ -\delta(0) \int_{-\infty}^{+\infty} dk^1 \ln \left(\cosh \vartheta(k^1, \varsigma) \right) \right\} \Rightarrow \quad (323)$$

$$\Rightarrow \exp \left\{ -\frac{\delta(0)}{\varsigma} \frac{\pi^2}{6} \right\} \Rightarrow \exp \left\{ -\frac{L}{\varsigma} \frac{\pi}{12} \right\} \xrightarrow{L \rightarrow \infty} 0, \quad (324)$$

also demonstrating the unitary inequivalence of corresponding representations [35].

5.2 Appendix B

The Wightman functions for zero and nonzero temperature (41) and (231), (233) admit the useful representations with different real parts, but with the same imaginary part, so that for: $\xi = \pm$, $k^0 = |k^1|$,

$$z^\xi = x^\xi - y^\xi, \quad z^\xi = z^0 + \xi z^1, \quad z^+ z^- = (z^0)^2 - (z^1)^2 \equiv z^2, \quad \text{and for any } A, B, \mathcal{F}(z) \text{ with} \quad (325)$$

$$\left(\mathcal{F}(z^{-\xi})\right)^A \left(\mathcal{F}(z^\xi)\right)^B = \left(\mathcal{F}(z^-)\right)^{(A+B)/2+\xi(A-B)/2} \left(\mathcal{F}(z^+)\right)^{(A+B)/2-\xi(A-B)/2} = \quad (326)$$

$$= \left(\mathcal{F}(z^-)\mathcal{F}(z^+)\right)^{(A+B)/2} \left(\frac{\mathcal{F}(z^-)}{\mathcal{F}(z^+)}\right)^{\xi(A-B)/2}, \quad \text{one has it as following:} \quad (327)$$

$$\left[\varphi^{\xi(+)}(x^\xi), \varphi^{\xi'(-)}(y^{\xi'})\right] = \frac{\delta_{\xi,\xi'}}{i} D^{(-)}(z^\xi), \quad (328)$$

$$\frac{1}{i} D^{(-)}(z^\xi) = \frac{1}{4\pi} \int_{\mu}^{\infty} \frac{d\lambda}{\lambda} e^{-i\lambda(z^\xi - i0)} = \frac{1}{4\pi} \int_{i\mu(z^\xi - i0)}^{\infty} e^{-t} d(\ln t) = -\frac{1}{4\pi} \ln \left(i\bar{\mu} \{z^\xi - i0\} \right) = \quad (329)$$

$$= -\frac{1}{4\pi} \ln \left| \bar{\mu} z^\xi \right| - \frac{i}{8} \varepsilon(z^\xi) = -\frac{1}{8\pi} \left[\ln |\bar{\mu}^2 z^2| + \xi \ln \left| \frac{z^+}{z^-} \right| \right] - \frac{i}{8} \left[\varepsilon(z^0) \theta(z^2) + \xi \varepsilon(z^1) \theta(-z^2) \right], \quad (330)$$

$$\left[\varphi^{\xi(+)}(x^\xi; [\pm]\varsigma), \varphi^{\xi'(-)}(y^{\xi'}; [\pm]\varsigma)\right] = \frac{\delta_{\xi,\xi'}}{4\pi} \int_{-\infty}^{\infty} \frac{dk^1}{k^0} \theta(-\xi k^1) \left[\cosh^2 \vartheta e^{-ik^0 z^\xi} + \sinh^2 \vartheta e^{ik^0 z^\xi} \right] = \quad (331)$$

$$= \frac{1}{4\pi} \delta_{\xi,\xi'} \left\{ \int_{\mu}^{\infty} \frac{dk^1}{k^0} e^{-ik^0(z^\xi - i0)} + \int_0^{\infty} \frac{dk^1}{k^0} \frac{2}{e^{\varsigma k^0} - 1} \left[\cos(k^0 z^\xi) - 1 \right] + \int_{\mu_1}^{\infty} \frac{dk^1}{k^0} \frac{2}{e^{\varsigma k^0} - 1} \right\} \equiv \quad (332)$$

$$\equiv \frac{\delta_{\xi,\xi'}}{i} \mathcal{D}^{(-)}(z^\xi, \varsigma; \mu_1) \equiv -\frac{1}{4\pi} \delta_{\xi,\xi'} \left\{ \ln \left(i\bar{\mu} \frac{\varsigma}{\pi} \sinh \left(\frac{\pi}{\varsigma} (z^\xi - i0) \right) \right) - g(\varsigma, \mu_1) \right\} = \quad (333)$$

$$= -\frac{1}{4\pi} \delta_{\xi,\xi'} \left\{ \ln \left| \bar{\mu} \frac{\varsigma}{\pi} \sinh \left(\frac{\pi}{\varsigma} z^\xi \right) \right| + i \frac{\pi}{2} \varepsilon(z^\xi) - g(\varsigma, \mu_1) \right\}, \quad \text{or:} \quad (334)$$

$$\frac{1}{i} \mathcal{D}^{(-)}(z^\xi; \varsigma; \mu_1) = -\frac{1}{8\pi} \left[\ln \left| \left(\bar{\mu} \frac{\varsigma}{\pi} \right)^2 \sinh \left(\frac{\pi}{\varsigma} z^+ \right) \sinh \left(\frac{\pi}{\varsigma} z^- \right) \right| + \xi \ln \left| \frac{\sinh(\pi z^+/\varsigma)}{\sinh(\pi z^-/\varsigma)} \right| - 2g(\varsigma, \mu_1) \right] - \frac{i}{8} \left[\varepsilon(z^0) \theta(z^2) + \xi \varepsilon(z^1) \theta(-z^2) \right] \Rightarrow -\frac{1}{4\pi} \left[\ln \left(i\bar{\mu} \{z^\xi - i0\} \right) - g(\varsigma, \mu_1) \right], \quad \text{for: } z^\xi \rightarrow 0, \quad (335)$$

$$\text{where: } \varepsilon(z^\xi) = \varepsilon(z^0) \theta(z^2) + \xi \varepsilon(z^1) \theta(-z^2), \quad \bar{\mu} = \mu e^{C_\Xi}, \quad C_\Xi = -\int_0^{\infty} dt e^{-t} \ln t, \quad (336)$$

$$\text{and: } \cosh^2 \vartheta = \frac{1}{1 - e^{-\varsigma k^0}} = 1 + \sinh^2 \vartheta, \quad \sinh^2 \vartheta = \frac{1}{e^{\varsigma k^0} - 1}, \quad \vartheta = \vartheta(k^1, \varsigma), \quad \text{so that:} \quad (337)$$

$$\left[\varphi^{\xi(\pm)}(x^\xi; [\pm]\varsigma), \tilde{\varphi}^{\xi'(\mp)}(y^{\xi'}; [\pm]\varsigma)\right] = (\pm 1)[\mp 1] \frac{1}{2\pi} \delta_{\xi,\xi'} \int_0^{\infty} \frac{dk^1}{k^0} \cosh \vartheta \sinh \vartheta \cos(k^0 z^\xi) = \quad (338)$$

$$= (\pm 1)[\pm 1] \frac{1}{4\pi} \delta_{\xi, \xi'} \left\{ \int_0^\infty \frac{dk^1}{k^0} \frac{1}{\sinh(\varsigma k^0/2)} [1 - \cos(k^0 z^\xi)] - \int_{\mu_2}^\infty \frac{dk^1}{k^0} \frac{1}{\sinh(\varsigma k^0/2)} \right\} = \quad (339)$$

$$= (\pm 1)[\pm 1] \frac{1}{4\pi} \delta_{\xi, \xi'} \left\{ \ln \left(\cosh \left(\frac{\pi}{\varsigma} z^\xi \right) \right) - f(\varsigma, \mu_2) \right\}, \quad (340)$$

$$\begin{aligned} [\varphi^{\xi(\pm)}(s; [\pm]\varsigma), Q^{\xi'(\mp)}([\pm]\varsigma)] &= \delta_{\xi, \xi'} \frac{i}{2} \int_{-\infty}^\infty dk^1 \theta(-\xi k^1) \left[e^{(\pm i)k^0(\hat{x}^0 - s)} \cosh^2 \vartheta - \right. \\ &\quad \left. - e^{(\mp i)k^0(\hat{x}^0 - s)} \sinh^2 \vartheta \right] \delta_L(k^1) = \delta_{\xi, \xi'} \frac{i}{2} \int_{-\infty}^\infty dk^1 \theta(-\xi k^1) \left[\cos(k^0(\hat{x}^0 - s)) (\cosh^2 \vartheta - \sinh^2 \vartheta) + \right. \\ &\quad \left. + (\pm 1)i \sin(k^0(\hat{x}^0 - s)) (\cosh^2 \vartheta + \sinh^2 \vartheta) \right] \delta_L(k^1) = \\ &= \delta_{\xi, \xi'} \frac{i}{4} \int_{-\infty}^\infty dk^1 \left[\cos(k^0(\hat{x}^0 - s)) + (\pm 1)i \sin(k^0(\hat{x}^0 - s)) \coth(\varsigma k^0/2) \right] \delta_L(k^1) \xrightarrow{L \rightarrow \infty} \\ &\xrightarrow{L \rightarrow \infty} \delta_{\xi, \xi'} \left[\frac{i}{4} - (\pm 1) \left(\frac{\hat{x}^0 - s}{2\varsigma} \right) \right], \end{aligned} \quad (341)$$

$$[\varphi^{\xi(\pm)}(s; [\pm]\varsigma), \tilde{Q}^{\xi'(\mp)}([\pm]\varsigma)] = (\pm 1)[\pm 1] \delta_{\xi, \xi'} \int_{-\infty}^\infty dk^1 \theta(-\xi k^1) \cosh \vartheta \sinh \vartheta \sin(k^0(\hat{x}^0 - s)) \delta_L(k^1) = \quad (342)$$

$$= (\pm 1)[\pm 1] \frac{\delta_{\xi, \xi'}}{2} \int_0^\infty dk^1 \frac{\sin(k^0(\hat{x}^0 - s))}{\sinh(\varsigma k^0/2)} \delta_L(k^1) \xrightarrow{L \rightarrow \infty} (\pm 1)[\pm 1] \delta_{\xi, \xi'} \left(\frac{\hat{x}^0 - s}{2\varsigma} \right), \quad (343)$$

with the same result for interchanged order of limit, $L \rightarrow \infty$, for example, for usual box:

$$\int_0^\infty dk^1 \frac{\sin(k^1(\hat{x}^0 - s))}{\sinh(\varsigma k^1/2)} \frac{\sin k^1 L}{k^1} = \int_{\hat{x}^0 - s - L}^{\hat{x}^0 - s + L} d\rho \int_0^\infty \frac{dk^1}{2} \frac{\sin(k^1 \rho)}{\sinh(\varsigma k^1/2)} = \frac{1}{2} \ln \left[\frac{\cosh(\pi(L + \hat{x}^0 - s)/\varsigma)}{\cosh(\pi(L - \hat{x}^0 + s)/\varsigma)} \right] \rightarrow \frac{\hat{x}^0 - s}{\varsigma}.$$

Here the formulas (353)–(358) are used (see also 3.951.(18),(19) from [65]).

The imaginary part of Wightman function (39), (40) is defined by the commutative function (35) [33]:

$$\frac{1}{i} D^{(-)}(z) = \int \frac{d^2 k}{2\pi} \theta(k^0) \delta(k^2) e^{-i(kz)} \Rightarrow \frac{(-1)}{4\pi} \ln(-\bar{\mu}^2 z^2 + i0 \varepsilon(z^0)), \quad \frac{1}{i} D^{(-)}(z) - \left(\frac{1}{i} D^{(-)}(z) \right)^* = \quad (344)$$

$$= \frac{1}{i} D_0(z) = \int \frac{d^2 k}{2\pi} \varepsilon(k^0) \delta(k^2) e^{-i(kz)} = \frac{1}{2i} \varepsilon(z^0) \theta(z^2) = \frac{1}{4i} [\varepsilon(z^\xi) + \varepsilon(z^{-\xi})] = \frac{1}{2i} [\theta(z^\xi) - \theta(-z^{-\xi})], \quad (345)$$

$$\text{with: } \frac{\partial}{\partial z^\nu} \varepsilon(z^\xi) = 2(\xi 1)^\nu \delta(z^\xi), \quad D_0(0) = 0, \quad \frac{\partial}{\partial z^0} D_0(z) \Big|_{z^0=0} = \delta(z^1). \quad (346)$$

The relations (39), (40) correspond to following expansion of the distribution [7]:

$$\theta(k^0) \delta(k^2) = \frac{\theta(k^\xi)}{k^\xi} \delta(k^{-\xi}) + \frac{\theta(k^{-\xi})}{k^{-\xi}} \delta(k^\xi), \quad k^\xi = k^0 + \xi k^1. \quad (347)$$

5.3 Appendix C

The asymptotic expansions (246), (248) are obtained by use of the known series [65] with Bernoulli numbers B_i , and Bernoulli polynomials $B_i(x)$, that give the asymptotic expansion for corresponding integral till the moments I_n^Δ (244) exist:

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \sum_{j=1}^{\infty} B_{2j} \frac{z^{2j}}{(2j)!}, \quad B_0 = 1, \quad B_1 = -\frac{1}{2}, \quad B_{2j+1} = 0, \quad j \geq 1, \quad |z| < 2\pi, \quad z = \frac{t\zeta}{L}, \quad (348)$$

$$\frac{z/2}{\sinh(z/2)} = 1 + \sum_{j=1}^{\infty} B_{2j} \left(\frac{1}{2}\right) \frac{z^{2j}}{(2j)!}, \quad B_{2j} \left(\frac{1}{2}\right) = \left(\frac{1}{2^{2j-1}} - 1\right) B_{2j}, \quad B_{2j} = 2(-1)^{j+1} \zeta(2j) \frac{(2j)!}{(2\pi)^{2j}}, \quad (349)$$

where $\zeta(s)$ - is Riemann zeta-function. Various meanings of non-negative values of $I_0^\Delta, I_1^\Delta, I_2^\Delta$ depend on the choice of the volume cut-off regularization function $\Delta(y^1/L)$ and are depicted below for the some popular examples [13, 33, 47]:

$\Delta(x^1/L)$	$e^{-(x^1/L)^2}$	$e^{- x^1/L }$	$(1 - x^1 /L)\theta(1 - x^1 /L)$	$\theta(1 - x^1 /L)$	[33]: $\epsilon_r \rightarrow 0, r \rightarrow \infty$
$\overline{\Delta}(t)$	$\frac{e^{-(t/2)^2}}{2\sqrt{\pi}}$	$\frac{1}{\pi(1+t^2)}$	$\frac{1}{2\pi} \frac{\sin^2(t/2)}{(t/2)^2}$	$\frac{1}{\pi} \frac{\sin t}{t}$	$\mathcal{T}_r^{\frac{1}{2}}(t) \frac{ t ^{\frac{1}{r}-1}}{2r} \theta(1 - t)$
$2\pi I_0^\Delta$	$\sqrt{2\pi}/4$	$1/2$	$1/3$	1	$0 \leq \lim_{r \rightarrow \infty} \frac{\pi \ln 2}{2r^2 \epsilon_r} \leq \infty$
$\pi I_1^\Delta = a_0$	$1/4$	$1/2\pi$	$\ln 2/\pi$	∞	0
πI_2^Δ	$\sqrt{2\pi}/8$	$1/4$	$1/2$	∞	0

Note that the Oksak regularization [33] in the last column is essentially of another type, which does not implies the thermodynamic limit at all, since it implies: $L \mapsto L_0 = \text{const}$, $\lim_{r \rightarrow \infty} L_0 \overline{\Delta}_r(k^1 L_0) = \delta(k^1)$. Here $\sqrt{\mathcal{T}_r}(t) \simeq \mathcal{T}_r(t)$ is a smooth ‘‘trapezium’’ - like function: $\mathcal{T}_r(t) = 1$, for $2\epsilon_r < t < 1 - 2\epsilon_r$, $\mathcal{T}_r(t) \neq 0$, for $\epsilon_r \leq t \leq 1 - \epsilon_r$, so that $\epsilon_r \mathcal{T}_r'(t) \simeq \pm 1$, when it is not equal to 0. The even regularization function $\Delta_r(x^1/L_0)$ may be found up to the terms of order $O(\epsilon_r)$, for $\delta_r = 1/r$, as:

$$\Delta_r(x^1/L_0) = \int_{-\infty}^{\infty} dk^1 L_0 \overline{\Delta}_r(k^1 L_0) e^{ik^1 x^1} \implies \int_0^1 d(t^{1/r}) \cos(t x^1/L_0) = \quad (350)$$

$$= \Gamma(1 + \delta_r) \left\{ \cos(x^1/L_0) \sum_{n=0}^{\infty} \frac{(-1)^n (x^1/L_0)^{2n}}{\Gamma(2n + 1 + \delta_r)} + \sin(x^1/L_0) \sum_{n=0}^{\infty} \frac{(-1)^n (x^1/L_0)^{2n+1}}{\Gamma(2n + 2 + \delta_r)} \right\}, \quad (351)$$

which oscillates and very slowly decrease for any finite r , but uniformly for $x^1 < \infty$ tends to 1 with $r \rightarrow \infty$:

$$\Delta_r(x^1/L_0) \implies \left(L_0/|x^1|\right)^{1/r} \Gamma(1 + \delta_r) \cos(\pi \delta_r/2), \quad \text{for: } |x^1| \rightarrow \infty, \quad \Delta_r(x^1/L_0) \xrightarrow[r \rightarrow \infty]{} 1. \quad (352)$$

Thus, the arbitrary fixed parameter L_0 has nothing to do with a thermodynamic parameter of effective box size. Nevertheless, the values of $a_{0,1,2}$ [33] can be defined by the same way (244)–(248) as above, with $a_0 = a_1 - a_2 \Rightarrow 0$, for $r \rightarrow \infty$, because for $n > 0$ all $I_n^\Delta = 0$ with $r \rightarrow \infty$, independently on the view of $\mathcal{T}_r(t)$ and ϵ_r . Only the value of $2\pi I_0^\Delta$ can be chosen as finite for $\epsilon_r = 1/r^2$, as shown in the Table.

The last but one column shows that the usual box needs a separate consideration, due to divergent values of integrals for a_0 (244) and a_1 for this case. This may be done with the help of known elementary series and integrals [65]:

$$\coth \pi y = \frac{1}{\pi y} + \frac{2y}{\pi} \sum_{n=0}^{\infty} \frac{1}{y^2 + (n+1/2)^2}, \quad \tanh \pi y = \frac{2y}{\pi} \sum_{n=0}^{\infty} \frac{1}{y^2 + (n+1/2)^2}, \quad (353)$$

$$2 \int_0^{\infty} dx \sin bx e^{-\lambda x} = \frac{2b}{b^2 + \lambda^2}, \quad \text{for } \lambda > 0 \quad \int_a^b d\rho \sin \rho x = \frac{\cos ax - \cos bx}{x}, \quad \text{that give:} \quad (354)$$

$$2 \int_0^{\infty} dx \frac{\sin bx}{e^{\lambda x} - 1} = \sum_{n=0}^{\infty} \frac{2b}{b^2 + \lambda^2(n+1)^2} = \frac{\pi}{\lambda} \coth \left(\frac{\pi b}{\lambda} \right) - \frac{1}{b} = \frac{d}{db} \ln \left[\frac{1}{b} \sinh \left(\frac{\pi b}{\lambda} \right) \right], \quad (355)$$

$$\int_0^{\infty} dx \frac{\sin bx}{\sinh(\lambda x/2)} = \sum_{n=0}^{\infty} \frac{2b}{b^2 + \lambda^2(n+1/2)^2} = \frac{\pi}{\lambda} \tanh \left(\frac{\pi b}{\lambda} \right) = \frac{d}{db} \ln \left[\cosh \left(\frac{\pi b}{\lambda} \right) \right], \quad (356)$$

$$2 \int_0^{\infty} \frac{dx}{x} \left(\frac{\cos ax - \cos bx}{e^{\lambda x} - 1} \right) = \ln \left[\frac{1}{b} \sinh \left(\frac{\pi b}{\lambda} \right) \right] - \ln \left[\frac{1}{a} \sinh \left(\frac{\pi a}{\lambda} \right) \right], \quad (357)$$

$$\int_0^{\infty} \frac{dx}{x} \left(\frac{\cos ax - \cos bx}{\sinh(\lambda x/2)} \right) = \ln \left[\cosh \left(\frac{\pi b}{\lambda} \right) \right] - \ln \left[\cosh \left(\frac{\pi a}{\lambda} \right) \right]. \quad (358)$$

The L - independence of a_0 implies the change of variable in Eq. (244), which becomes impossible for divergent integral. Introducing again the ultraviolet cut-off $\Lambda = 1/\rho$, following to (317), the finite expressions for the values of a_0 , a_1 , a_2 , by virtue of (354)–(358), reads:

$$a_0^{reg}(L) \Rightarrow \frac{1}{\pi} \int_0^{\infty} \frac{dk^1}{k^1} \sin^2(k^1 L) e^{-k^1 \rho} = \int_0^L d\ell \frac{da_0^{reg}(\ell)}{d\ell} = \frac{1}{4\pi} \ln \left(\frac{(2L)^2 + \rho^2}{\rho^2} \right) \xrightarrow{L \rightarrow \infty} \frac{1}{2\pi} \ln \left(\frac{2L}{\rho} \right), \quad (359)$$

$$a_1^{reg}(L) \equiv \pi \int_0^{\infty} dk^1 k^1 \left(L \overline{\Delta}(k^1 L) \right)^2 \coth \left(\frac{k^0 \varsigma}{2} \right) \Rightarrow \frac{1}{\pi} \int_0^{\infty} \frac{dk^1}{k^1} \sin^2(k^1 L) e^{-k^1 \rho} \coth \left(\frac{k^0 \varsigma}{2} \right) \Rightarrow \quad (360)$$

$$\Rightarrow a_0^{reg}(L) + \frac{1}{2\pi} \ln \left[\frac{\varsigma}{2\pi L} \sinh \left(\frac{2\pi L}{\varsigma} \right) \right] \xrightarrow{L \rightarrow \infty} \frac{L}{\varsigma} - \frac{1}{2\pi} \ln 2 - \frac{1}{2\pi} \ln \left(\frac{\pi \rho}{\varsigma} \right), \quad (361)$$

$$a_2(L) \equiv \pi \int_0^{\infty} dk^1 k^1 \frac{\left(L \overline{\Delta}(k^1 L) \right)^2}{\sinh(k^0 \varsigma/2)} \Rightarrow \frac{1}{\pi} \int_0^{\infty} \frac{dk^1}{k^1} \frac{\sin^2(k^1 L)}{\sinh(k^0 \varsigma/2)} = \frac{1}{2\pi} \ln \left[\cosh \left(\frac{2\pi L}{\varsigma} \right) \right] \xrightarrow{L \rightarrow \infty} \frac{L}{\varsigma} - \frac{1}{2\pi} \ln 2, \quad (362)$$

$$\text{so that: } a_1^{reg}(L) - a_2(L) \xrightarrow{L \rightarrow \infty} -\frac{1}{2\pi} \ln \left(\frac{\pi \rho}{\varsigma} \right) = \frac{1}{2\pi} \ln \left(\frac{\Lambda}{\pi k_B T} \right). \quad (363)$$

Thus, all the L - dependence again is canceled exactly, as well as for the convergent case, and in accordance with the Table, the finite value of $2\pi I_0^\Delta = 1$, while the remaining difference (363) may be associated with zero only for high temperature ($\varsigma \rightarrow 0$) by choosing the ultraviolet cut-off as $\Lambda = \pi k_B T$.

The Table obviously demonstrate that asymptotic of Fourier image $\overline{\Delta}(t)$ in fact is defined by smoothness of the original $\Delta(x^1/L)$. For its discontinuous derivative of n -th order: $\overline{\Delta}(t) \sim t^{-1-n}$. Thus, $I_{0,1,2}^\Delta$ exist for $2n > 1$, for example, for continuous function with first derivative discontinuous at finite number of points.

5.4 Appendix D

From (280), by introducing the parameter z for additional control, as:

$$F_\eta([\pm]\varsigma) = \eta \int_0^\infty \frac{dk^1}{4\pi k^0} \left[c(k^1; [\pm]\varsigma) c(-k^1; [\pm]\varsigma) - c^\dagger(k^1; [\pm]\varsigma) c^\dagger(-k^1; [\pm]\varsigma) \right] + \\ + z\eta \int_0^\infty \frac{dk^1}{4\pi k^0} \left[\tilde{c}(k^1; [\pm]\varsigma) \tilde{c}(-k^1; [\pm]\varsigma) - \tilde{c}^\dagger(k^1; [\pm]\varsigma) \tilde{c}^\dagger(-k^1; [\pm]\varsigma) \right],$$

one has the following nonzero contributions:

$$\begin{aligned} & \left[F_\eta([\pm]\varsigma), \varphi^{\xi(+)}(s; [\pm]\varsigma) \right] = \\ & = \eta \int_0^\infty \frac{dk^1}{4\pi k^0} \left(\frac{\xi}{2\pi} \right) \int_{-\infty}^\infty \frac{dq^1}{2q^0} \theta(-\xi q^1) \cosh \vartheta e^{-iq^0 s} \left[c^\dagger(k^1; [\pm]\varsigma) c^\dagger(-k^1; [\pm]\varsigma), c(q^1) \right] - \\ & - z\eta \int_0^\infty \frac{dk^1}{4\pi k^0} \left(\frac{\xi}{2\pi} \right) \int_{-\infty}^\infty \frac{dq^1}{2q^0} \theta(-\xi q^1) \cosh \vartheta e^{-iq^0 s} \left[\tilde{c}(k^1; [\pm]\varsigma) \tilde{c}(-k^1; [\pm]\varsigma), c(q^1) \right] + \\ & + \eta \int_0^\infty \frac{dk^1}{4\pi k^0} \left([\pm 1] \frac{\xi}{2\pi} \right) \int_{-\infty}^\infty \frac{dq^1}{2q^0} \theta(-\xi q^1) \sinh \vartheta e^{+iq^0 s} \left[c(k^1; [\pm]\varsigma) c(-k^1; [\pm]\varsigma), \tilde{c}(q^1) \right] - \\ & - z\eta \int_0^\infty \frac{dk^1}{4\pi k^0} \left([\pm 1] \frac{\xi}{2\pi} \right) \int_{-\infty}^\infty \frac{dq^1}{2q^0} \theta(-\xi q^1) \sinh \vartheta e^{iq^0 s} \left[\tilde{c}^\dagger(k^1; [\pm]\varsigma) \tilde{c}^\dagger(-k^1; [\pm]\varsigma), \tilde{c}(q^1) \right]. \end{aligned}$$

By virtue of (185) it is a simple matter to see that:

$$\begin{aligned} & \left[c^\dagger(k^1; [\pm]\varsigma), c(q^1) \right] = \left[\tilde{c}^\dagger(k^1; [\pm]\varsigma), \tilde{c}(q^1) \right] = -(2\pi) (2k^0) \cosh \vartheta \delta(k^1 - q^1), \\ & \left[\tilde{c}(k^1; [\pm]\varsigma), c(q^1) \right] = \left[c(k^1; [\pm]\varsigma), \tilde{c}(q^1) \right] = [\pm 1] (2\pi) (2k^0) \sinh \vartheta \delta(k^1 - q^1), \end{aligned}$$

what leads to:

$$\begin{aligned} & \left[c^\dagger(k^1; [\pm]\varsigma) c^\dagger(-k^1; [\pm]\varsigma), c(q^1) \right] = -4\pi k^0 \cosh \vartheta \left[\delta(k^1 + q^1) c^\dagger(k^1; [\pm]\varsigma) + \delta(k^1 - q^1) c^\dagger(-k^1; [\pm]\varsigma) \right], \\ & \left[\tilde{c}(k^1; [\pm]\varsigma) \tilde{c}(-k^1; [\pm]\varsigma), c(q^1) \right] = [\pm 1] 4\pi k^0 \sinh \vartheta \left[\delta(k^1 + q^1) \tilde{c}(k^1; [\pm]\varsigma) + \delta(k^1 - q^1) \tilde{c}(-k^1; [\pm]\varsigma) \right], \\ & \left[c(k^1; [\pm]\varsigma) c(-k^1; [\pm]\varsigma), \tilde{c}(q^1) \right] = [\pm 1] 4\pi k^0 \sinh \vartheta \left[\delta(k^1 + q^1) c(k^1; [\pm]\varsigma) + \delta(k^1 - q^1) c(-k^1; [\pm]\varsigma) \right], \\ & \left[\tilde{c}^\dagger(k^1; [\pm]\varsigma) \tilde{c}^\dagger(-k^1; [\pm]\varsigma), \tilde{c}(q^1) \right] = -4\pi k^0 \cosh \vartheta \left[\delta(k^1 + q^1) \tilde{c}^\dagger(k^1; [\pm]\varsigma) + \delta(k^1 - q^1) \tilde{c}^\dagger(-k^1; [\pm]\varsigma) \right]. \end{aligned}$$

Substitution of these expressions gives:

$$\begin{aligned}
& \left[F_\eta([\pm]\varsigma), \varphi^{\xi(+)}(s; [\pm]\varsigma) \right] = \\
& = \frac{\xi\eta}{2\pi} \int_{-\infty}^{\infty} \frac{dq^1}{2q^0} \theta(-\xi q^1) \cosh \vartheta e^{-iq^0 s} \left(-\cosh \vartheta c^\dagger(-q^1; [\pm]\varsigma) - [\pm 1] z \sinh \vartheta \tilde{c}(-q^1; [\pm]\varsigma) \right) + \\
& + \frac{\xi\eta}{2\pi} \int_{-\infty}^{\infty} \frac{dq^1}{2q^0} \theta(-\xi q^1) \sinh \vartheta e^{+iq^0 s} \left(\sinh \vartheta c(-q^1; [\pm]\varsigma) + [\pm 1] z \cosh \vartheta \tilde{c}^\dagger(-q^1; [\pm]\varsigma) \right),
\end{aligned}$$

where:

$$\begin{aligned}
& -\cosh \vartheta c^\dagger(-q^1; [\pm]\varsigma) - [\pm 1] z \sinh \vartheta \tilde{c}(-q^1; [\pm]\varsigma) = \\
& = -\left[\cosh^2 \vartheta - z \sinh^2 \vartheta \right] c^\dagger(-q^1) + [\pm 1] [1 - z] \sinh \vartheta \cosh \vartheta \tilde{c}(-q^1) \implies -c^\dagger(-q^1), \\
& \sinh \vartheta c(-q^1; [\pm]\varsigma) + [\pm 1] z \cosh \vartheta \tilde{c}^\dagger(-q^1; [\pm]\varsigma) = \\
& = -[\pm 1] \left[\sinh^2 \vartheta - z \cosh^2 \vartheta \right] \tilde{c}^\dagger(-q^1) + [1 - z] \sinh \vartheta \cosh \vartheta c(-q^1) \implies [\pm 1] \tilde{c}^\dagger(-q^1),
\end{aligned}$$

for $z = 1$. Thus, due to (222), (223):

$$\begin{aligned}
& \left[\varphi^{\xi(+)}(s; [\pm]\varsigma), F_\eta([\pm]\varsigma) \right] = \frac{\xi\eta}{2\pi} \int_{-\infty}^{\infty} \frac{dq^1}{2q^0} \theta(-\xi q^1) \cosh \vartheta e^{-iq^0 s} c^\dagger(-q^1) - \\
& - [\pm 1] \frac{\xi\eta}{2\pi} \int_{-\infty}^{\infty} \frac{dq^1}{2q^0} \theta(-\xi q^1) \sinh \vartheta e^{+iq^0 s} \tilde{c}^\dagger(-q^1) \equiv \eta \varphi^{-\xi(-)}(-s; [\pm]\varsigma).
\end{aligned}$$

The next commutator in (283) is obtained by the same way.

5.5 Appendix E

The free fermionic annihilation/creation operators: $\{b(p^1), b^\dagger(q^1)\} = \{f(p^1), f^\dagger(q^1)\} = \delta(p^1 - q^1)$ with correct parity properties: $\mathcal{P}b^\#(p^1)\mathcal{P}^{-1} = b^\#(-p^1)$, $\mathcal{P}f^\#(p^1)\mathcal{P}^{-1} = -f^\#(-p^1)$, for $\mathcal{P}\chi(x^1, x^0)\mathcal{P}^{-1} = \gamma^0\chi(-x^1, x^0)$, are defined in [13, 14] by following decompositions for the field $\chi(x)$, $\xi = \pm$:

$$\chi_\xi(x) = \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{2\pi}} \left[\theta(\xi p^1) b(p^1) e^{-i(p x)} + \xi \theta(\xi p^1) f^\dagger(p^1) e^{i(p x)} \right] e^{i\varpi - i\xi\Theta/4} \Rightarrow \chi_\xi(x^{-\xi}), \quad (364)$$

where: $(px) \Rightarrow p^0 x^{-\xi}$, for: $\xi p^1 = p^0 = |p^1|$, and initial overall and relative phases ϖ and Θ are introduced;

$$(\overline{\chi}(x))_\xi = \left(\chi^\dagger(x) \gamma^0 \right)_\xi = \chi_{-\xi}^\dagger(x^\xi) = \int_{-\infty}^{\infty} \frac{dp^1}{\sqrt{2\pi}} \left[\theta(-\xi p^1) b^\dagger(p^1) e^{i(p x)} - \xi \theta(-\xi p^1) f(p^1) e^{-i(p x)} \right] e^{-i\varpi - i\xi\Theta/4}, \quad (365)$$

where: $(px) \Rightarrow p^0 x^\xi$, for: $-\xi p^1 = p^0 = |p^1|$. The corresponding conserved charges are defined by components of the vector current : $J_{(\chi)}^\nu(x) := \bar{\chi}(x) \gamma^\nu \chi(x) := \chi^\dagger(x) \gamma^0 \gamma^\nu \chi(x)$:, as [33, 60, 61]:

$$\frac{O_{(\chi)}}{\sqrt{\pi}} = \int_{-\infty}^{\infty} dx^1 : J_{(\chi)}^0(x) := \int_{-\infty}^{\infty} dx^1 : \chi^\dagger(x) \chi(x) := \int_{-\infty}^{\infty} dx^1 \sum_{\xi=\pm} : \chi_\xi^\dagger(x^{-\xi}) \chi_\xi(x^{-\xi}) : , \quad (366)$$

$$\frac{O_{5(\chi)}}{\sqrt{\pi}} = \int_{-\infty}^{\infty} dx^1 : J_{(\chi)}^1(x) := \int_{-\infty}^{\infty} dx^1 : \chi^\dagger(x) \gamma^5 \chi(x) := \int_{-\infty}^{\infty} dx^1 \sum_{\xi=\pm} \xi : \chi_\xi^\dagger(x^{-\xi}) \chi_\xi(x^{-\xi}) : , \quad \text{where:} \quad (367)$$

$$\frac{Q^{-\xi}}{\sqrt{\pi}} = \frac{1}{2\sqrt{\pi}} [O_{(\chi)} + \xi O_{5(\chi)}] = \int_{-\infty}^{\infty} dx^1 : \chi_\xi^\dagger(x) \chi_\xi(x) := \int_{-\infty}^{\infty} dp^1 \theta(\xi p^1) [b^\dagger(p^1) b(p^1) - f^\dagger(p^1) f(p^1)] , \quad (368)$$

recasts them into: $\frac{O_{(\chi)}}{\sqrt{\pi}} = \int_{-\infty}^{\infty} dp^1 [b^\dagger(p^1) b(p^1) - f^\dagger(p^1) f(p^1)]$, with: $\theta(\xi p^1) + \theta(-\xi p^1) = 1$, (369)

$$\frac{O_{5(\chi)}}{\sqrt{\pi}} = \int_{-\infty}^{\infty} dp^1 \varepsilon(p^1) [b^\dagger(p^1) b(p^1) - f^\dagger(p^1) f(p^1)] , \quad \text{with:} \quad \xi [\theta(\xi p^1) - \theta(-\xi p^1)] = \varepsilon(p^1). \quad (370)$$

Thus: $\frac{Q^-}{\sqrt{\pi}} = Q_R = N_R^+ - N_R^-$, $\frac{Q^+}{\sqrt{\pi}} = Q_L = N_L^+ - N_L^-$, $N_{(F)}^\pm = N_R^\pm + N_L^\pm$, (371)

$$\frac{O_{(\chi)}}{\sqrt{\pi}} = Q_{(F)} = Q_R + Q_L = N_{(F)}^+ - N_{(F)}^- , \quad \frac{O_{5(\chi)}}{\sqrt{\pi}} = Q_{5(F)} = Q_R - Q_L. \quad (372)$$

The normal form of free Hamiltonian (7) reads:

$$: H_{0[\chi]}(x^0) := \int_{-\infty}^{\infty} dx^1 : \chi^\dagger(x) \gamma^5 (-i \partial_1) \chi(x) := \int_{-\infty}^{\infty} dp^1 |p^1| [b^\dagger(p^1) b(p^1) + f^\dagger(p^1) f(p^1)] . \quad (373)$$

The expressions (160)–(162) are obtained following to [31], but with $\eta_{(F)} = +1$, $\eta_{(B)} = -1$ and energy level density $\mathcal{D}(\epsilon, L) \Rightarrow g_s^{(\eta)} 2L/(hc)$ for the spectrum like (373), $\epsilon(p^1) = c|p^1|$ and spin degeneracy $g_s^{(\eta)}$, from general definitions [59] with chemical potentials μ_η and $\gamma_\eta = \zeta \mu_\eta = \mu_\eta/(k_B T)$:

$$N_\eta = L \bar{n}_\eta = \int_0^\infty d\epsilon \langle \langle n_\eta(\epsilon, \mu) \rangle \rangle \mathcal{D}(\epsilon, L) \Rightarrow \frac{g_s^{(\eta)} 2L}{\zeta hc} \mathcal{F}_0^{(\eta)}(\gamma_\eta) = \frac{g_s^{(\eta)} 2L}{\zeta hc} \eta \ln(1 + \eta e^{\gamma_\eta}) , \quad (374)$$

$$P_\eta L = \int_0^\infty d\epsilon \langle \langle n_\eta(\epsilon, \mu) \rangle \rangle \int_0^\epsilon d\epsilon' \mathcal{D}(\epsilon', L) \Rightarrow \frac{g_s^{(\eta)} 2L}{\zeta^2 hc} \mathcal{F}_1^{(\eta)}(\gamma_\eta) , \quad 2\mathcal{F}_1^{(1)}(0) = \mathcal{F}_1^{(-1)}(0) = \frac{\pi^2}{6} , \quad (375)$$

with: $\langle \langle n_\eta(\epsilon, \mu) \rangle \rangle = \frac{1}{e^{\zeta(\epsilon-\mu)} + \eta}$, $\mathcal{F}_\lambda^{(\eta)}(\gamma) = \int_0^\infty dx \frac{x^\lambda}{e^{x-\gamma} + \eta}$, $\frac{d\mathcal{F}_\lambda^{(\eta)}(\gamma)}{d\gamma} = \lambda \mathcal{F}_{\lambda-1}^{(\eta)}(\gamma)$, (376)

$$\frac{d}{d\gamma} (\mathcal{F}_1^{(1)}(\gamma) + \mathcal{F}_1^{(1)}(-\gamma)) = \mathcal{F}_0^{(1)}(\gamma) - \mathcal{F}_0^{(1)}(-\gamma) = \gamma , \quad \mathcal{F}_1^{(1)}(\gamma) + \mathcal{F}_1^{(1)}(-\gamma) = 2\mathcal{F}_1^{(1)}(0) + \frac{\gamma^2}{2} . \quad (377)$$

Eqs. (160) are given by Eq. (374) for $N_\eta = N_{(B)}$ bose particles and for $N_\eta = N_{(F)}^\pm$ fermions and antifermions, pairing as $\chi^+ + \chi^- \rightleftharpoons$ to arbitrary number of these bosons. Since $\gamma_{(B)} = 0$, then for equilibrium $\gamma_{(F)}^+ + \gamma_{(F)}^- \Rightarrow 0$, what transcribes the Eqs. (375), (377) with $g_s^{(1,\pm)} = g_s^{(-1)} = 1$, as Eq.(162) for total charge, and Eq. (161) for total pressure $P_{(F)} = P_{(F)}^+ + P_{(F)}^-$, with internal energy density:

$$\frac{\mathcal{U}_{(F)}}{L} \equiv - \left[\left(\varsigma \frac{\partial P_{(F)}}{\partial \varsigma} \right)_\gamma + P_{(F)} \right] = \frac{1}{\varsigma^2 \hbar c} \left(\frac{\pi^2}{3} + \gamma_{(F)}^2 \right) = P_{(F)}. \quad (378)$$

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